March 13 Math 103 212

Last Time: Separation of Variables

Ch 9 Differential Equations

1. Model with differential eqs.
2. Find steady state solutions/ stable or unstable/ when inc/dec-
3. Solve with separation of variables.

Definitions

(Def) An equilibrium / stable solution /

Steady state solution is a solution

to a differential equation which is constant for all t, \[ \frac{dy}{dt} = 0 \]

Ex of Diff eq. \[ \frac{dy}{dt} = y(t-2) \] separable

\[ \frac{dy}{dt} = y + t \] not separable

\[ \frac{dy}{dt} = 1 - y \] separable and autonomous
\[ \frac{dy}{dt} = y(t-2) = 0 \]

\[ y = 0 \] steady state solution

\[ \frac{dy}{dt} = 1 - y = 0 \]

\[ y = 1 \] steady state solution

Graph of \( \frac{dy}{dt} = 1 - y \)

Graph of \( \frac{dy}{dt} = y(t-2) \)

Phase diagram:

Graph of \( y \) vs. \( t \):

All steady state solutions.

(Stable solution.)
\[ \text{(Def)} \text{ we call an equilibrium} \]

- Stable if \[ \frac{\text{df}}{\text{dx}} = y(y-1)^2(y+2) \] have?
- Unstable if \[ \frac{\text{df}}{\text{dx}} = y(y-1)^2(y+2) \] have?
- Neither/semi-stable /Saddle

Clicker #1
How many \underline{stable} equilibrium

Sketch \( \frac{\text{dy}}{\text{dx}} \) vs \( y \)

\[ g(y) = y(y-1)^2(y+2) \]

1 stable solution.
Ex. (Population Growth/Decay)

The population grows at a rate proportional to its size.

\[ \frac{dp}{dt} = k \cdot P \]

Solve:

\[ \int \frac{1}{P} dp = \int k dt \]

\[ \ln|P| = kt + C_1 \]

\[ e^{kt+C_1} = |P| \]

Population is positive.

\[ e^{kt+C_1} = P \]

\[ e^{kt} e^{c_1} = P \]

\[ C_2 e^{kt} = P \]

Use \( P(0) = P_0 \)

\[ C_2 e^{0} = P_0 \]

\[ C_2 = P_0 \]
\[ P(t) = P_0 e^{kt} \]

\[ g(P) = kP \]

\[ \frac{dP}{dt} = g(P) = kP \]

**If a population doubles every year from births but has a 25% mortality rate, what differential equation models this?**

\[ \frac{dP}{dt} = \text{doubles from birth} - 25\% \text{ mortality rate} \]

\[ = 2P - \frac{1}{4}P \]

\[ = \frac{7}{4}P. \]
\[ \frac{dP}{dt} = \frac{K}{K-P} \]

**Solution**

\[ \int \frac{K}{K-P} \, dp = \int dt \]

\[ \ln |K-P| = K \cdot t + C \]

**Population growth**

Proportional to its size times the difference from capacity.
\[ S \frac{1}{P} + \frac{1}{K-P} \, dP = \alpha \, dt + C, \]

\[ \ln |P| + (-\ln |K-P|) = \alpha t + C, \]

\[ \ln \left| \frac{P}{K-P} \right| = \alpha t + C, \]

\[ e^{\alpha t + C} = \left| \frac{P}{K-P} \right| \]

**Assume** \( P_0 < K \) so \( K-P_0 > 0 \)

[Note: if instead \( P_0 > K \) \( K-P_0 < 0 \) so \( |K-P| = -(K-P) \)]

\[ C_2 \, e^0 = \frac{P_0}{K-P_0} \]

\[ C_2 = \frac{P_0}{K-P_0} \]

\[ \frac{P_0}{K-P_0} \, e^{\alpha t} = \frac{P}{K-P} \]

\[ P(t) = \left( \frac{K}{P_0} \right) e^{-\alpha t} + 1 \]
\[ g(p) = \alpha \frac{d}{dP} \left( \frac{K - P}{K} \right) \]

we can assume \( \alpha > 0 \).

\[ \begin{array}{c}
\text{unstable, stable}
\end{array} \]

\[ \begin{array}{c}
F (\text{Newton's Law of Cooling})
\end{array} \]

the rate of change of temperature is proportional to the difference between the temperature and the surrounding temp (Ts).

\[ \frac{dT}{dt} = \alpha (T_s - T) \]

assume positive.
\[ g(T) = \alpha (T_s - T) \]

Stable,

Clicker #3 (come back when the time hits 0 min)

Tea 100°C is in a room 20°C

A) What is the temperature at time t?

B) Assuming the proportionality constant for water is 0.000256
   How long until the tea is drinkable at 70°C?

A) Solve \[ \frac{dT}{dt} = k(20 - T) \]
   \[ T(0) = 100 \]
   \[ \int \frac{1}{20 - T} \, dt = \int k \, dt \]

\[ -\ln|20 - T| = kt + C_1 \]
\[ \ln|20 - T| = -kt + C_2 \]
\[ e^{-kt + C_2} = 20 - T \]
\[ C_3 e^{-kt} = 20 - 100 \]
\[ C_3 e^0 = 20 - 100 \]
\( a = -80 \)

\[-80 e^{-kt} = 20 - 1 \]

\[ T(t) = 20 + 80 e^{-kt} \]

\[ a + t = 0 \]
\[ T(0) = 100 \]
\[ t \to \infty \]
\[ T \to 20 \]

**Example (Modeling Infectious Diseases)**

\[ S + I = 1 \]
\[ 100\% \]
\[ S = 1 - I \]

\[ \frac{dS}{dt} = -\alpha SI \]

\[ \frac{dI}{dt} = \alpha SI \]

\[ \frac{dI}{dt} = \alpha I(1 - I) \]
\[
g'(t) = aI(1-I) \\
\text{All the pop. will be infected.} \\
\frac{dI}{dt} = aIS - \beta I \\
= aI(1-I) - \beta I \\
= aI - aI^2 - \beta I \\
= I \left( a - \beta - aI \right) \\
\text{Zeros: } 0, \quad \frac{a - \beta}{a} = 1 - \frac{\beta}{a} \\
1 - \frac{\beta}{a} < 0 \quad \Rightarrow \quad a I < 1 \quad \text{less than 1}
disease dies out.

% of pop is always sick.