§3.7 Sketching the graph of $\int_a^x f(t)\,dt$ using graph of $f(x)$

- Qualitative relation between $\int_a^x f(t)\,dt$ and $f(x)$

Recall:

This is a function of the variable x!

Signed Area $\int_a^x f(t)\,dt$

- the FTC: $\dfrac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$.

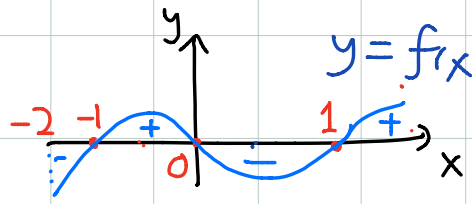
- If $f(x) \geq 0$ at x, ($f(x) > 0$)

then $\int_a^x f(t)\,dt$ ← function of x. is increasing (strictly) at x.

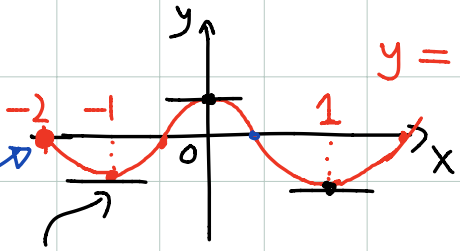
- If $f(x) \leq 0$ at x, ($f(x) < 0$)

then $\int_a^x f(t)\,dt$ ← function of x. is decreasing (strictly) at x.

e.g.



zeros of $f(x)$ are $x = -1, 0, 1$.



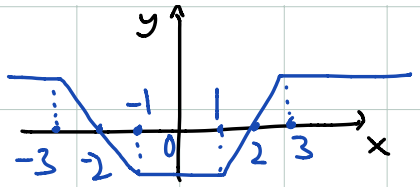
* $\int_{-2}^{-2} f(t)\,dt = 0$

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
$f(x)$	−	+	−	+
tangent line	↘	↗	↘	↗
$\int_a^x f(t)\,dt$	decreasing	increasing	decreasing	increasing

local min at $x = -1$
slope = 0
horizontal tangent

local max at $x = 0$
slope = 0
horizontal tangent

local min at $x = 1$
slope = 0
horizontal tangent

EX Let: 

$y = f(x)$.

Find where

- Local max. of $\int_0^x f(t)\,dt$
- Local min. of $\int_0^x f(t)\,dt$.

<sol> Note $\frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = f(x)$.

Let $G(x) = \int_0^x f(t)\,dt$.

Then $G'(x) = f(x)$.

	$x<-2$	$-2<x<2$	$x>2$
$G'(x)=f(x)$	+	−	+
slope of $G(x)$	↗	↘	↗
$G(x)$	increasing	decreasing	increasing

↑ local max at $x=-2$

↑ local min. at $x=2$

▨

Today

§4.1, §4.2,

Ch. 4 Velocities & Rates. (Rate of change & Net change).

Recall. Fundamental Theorem of Calculus.

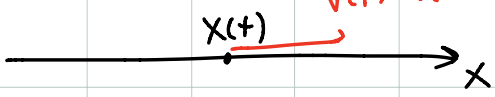
(Assuming { 1. the function $F(x)$ is differentiable. 2. the function $f(x)$ is continuous.)

- $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.
- $\int_a^x F'(t)\,dt = F(x) - F(a)$. ← Net change of F

($F'(t)$: derivative of F.)

§4.2.

a particle motion in t:

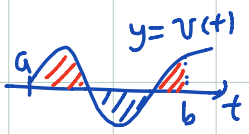


- position at time t: $X(t)$
- velocity at time t: $V(t) = X'(t)$
- acceleration at time t: $a(t) = V'(t) = X''(t)$.

displacement: Net change of position.

$X(b) - X(a)$ = displacement from $t=a$ to $t=b$.

$= \int_a^b X'(t)\,dt$ ($X'(t) = V(t)$)



Ex. A particle moves with velocity $v(t) = t-1$.

Suppose at $t=0$, the particle is located at $x=1$.

① Find the particle location at $t=2$.

② Find the total distance traveled from $t=0$ to $t=2$.

<sol>:

① the location at $t=2$ — the location at $t=0$.

$$\underbrace{X(2)-X(0)}_{\text{displacement from } t=0 \text{ to } t=2.} = \int_0^2 \frac{dX}{dt}\,dt$$

$$= \int_0^2 v(t)\,dt \leftarrow v(t) = \frac{dx}{dt}.$$

$$= \int_0^2 (t-1)\,dt$$

$$= \left[\frac{t^2}{2} - t\right]_0^2 \leftarrow \left(\frac{t^2}{2} - t\right)' = t-1.$$

$$= \left(\frac{2^2}{2} - 2\right) - \left(\frac{0}{2} - 0\right) = 0.$$

$$\therefore X(2) = X(0) + 0$$

$$= 1 + 0 = 1 \quad ▨$$

$$\boxed{X(b) = X(a) + \int_a^b X'(t)\,dt}$$

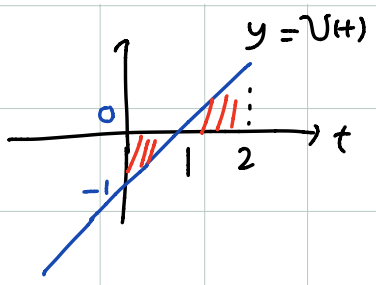
② Total distance traveled

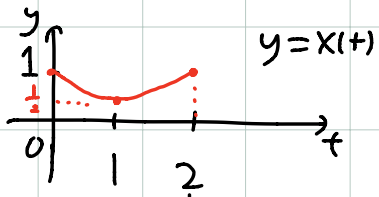
~~$|X(2) - X(1)|$~~

$= |1 - 1|$

WRONG.

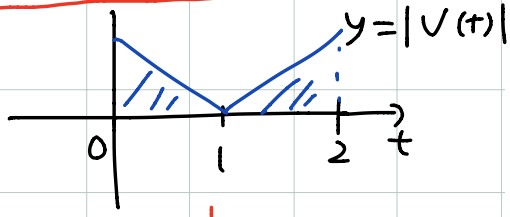
$= 0$





	$t<1$,	$1<t$
$V(t) = X'(t)$	$-$	$+$
$X(t)$	decreasing "backward move"	increasing "forward move"

Total traveled distance between $t=a$ & $t=b$ $= \int_a^b |X'(t)|\,dt \left(= \int_a^b |V(t)|\,dt\right)$



∴ Total traveled distance from $t=0$ to $t=2$

$= \int_0^2 |V(t)|\,dt$ ← Note the absolute value sign.

$|V(t)|$ = speed = absolute value of velocity

$= \int_0^2 |t-1|\,dt$

$= \int_0^1 -(t-1)\,dt + \int_1^2 (t-1)\,dt$

$= \frac{1}{2} + \frac{1}{2} = 1$ //

Ex. A particle motion (human, airplane, car, etc) : position $X(t)$ at time t.

acceleration $a(t)$ ($\leftarrow$ due to external force, e.g. gravity)

Suppose
- $a(t) = -2t$.
- initial position $X(0) = 3$.
- initial velocity $X'(0) = 2$

Find $X(t)$.

<sol> $X(t) = \underbrace{X(0)}_{=3} + \int_0^t X'(s)\,ds$

(used s because t is already used to denote a given time arbitrary)

To find $X'(s)$, consider

$X'(t) = \underbrace{X'(0)}_{=2} + \int_0^t \underbrace{X''(s)}_{\text{acceleration } a(s) = -2s}\,ds$

$= 2 + \int_0^t -2s\,ds$

$\left(\frac{d}{ds}\left(\frac{s^2}{2}\right) = -2s\right)$

$= 2 + \left[-2\cdot\frac{s^2}{2}\right]_0^t = 2 + \left[-2\cdot\frac{t^2}{2} - \left(-2\cdot\frac{0^2}{2}\right)\right]$

$= 2 - t^2$.

Therefore, $X(t) = 3 + \int_0^t (2 - s^2)\,ds = 3 + \left[2s - \frac{s^3}{3}\right]_0^t$

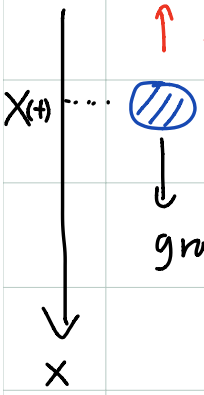
$= \underbrace{3 + 2t - \frac{t^3}{3}}$. ▨

$\left((2s)' = 2,\ \left(\frac{s^3}{3}\right)' = s^2\right)$

$$X'(b) = X'(a) + \int_a^b X''(s)\,ds$$

§ 4.2.4.

Falling body. $v(t)$: (vertical) velocity at time t.



$$\frac{dv}{dt} = g - kv$$

gravity; friction (it is proportional to the velocity.)

acceleration.

Ex $X(t)$: position at time t

$v(t)$: velocity

Suppose $X(0)=1$, $\boxed{\frac{dv}{dt} = 3 - v}$, and $v(0)=0$.

Find $X(t)$.

<sol>. Need to find $v(t)$. (Then $X(t) = X(0) + \int_0^t v(s)\,ds$)

Try

$$v(t) = v(0) + \int_0^t v'(s)\,ds$$

$$= v(0) + \int_0^t (3 - v(s))\,ds$$

This does not work. ??? we don't have an explicit expression for v, yet

Instead of this,

• Can consider $\frac{d^2}{dt^2} v = \frac{d}{dt}(3 - v) = -\frac{d}{dt} v$

$\frac{d}{dt} a(t)$ $\quad$ $a(t)$.

• $\therefore \quad a'(t) = -a(t)$.

So $a(t) = Ce^{-t}$ ← the acceleration.

This is a simple Differential Equation

$y'(t) = k\,y(t)$

$\Rightarrow y(t) = C \cdot e^{kt}$.

• To determine C.

use $a(0) = \frac{d}{dt}v\Big|_{t=0} = 3 - v(0)$

$= 0$ ← assumption $v(0)=0$

$= 3$.

From $a(0) = C \cdot e^{-0} \quad C = 3$.

Thus, $a(t) = 3e^{-t}$.

• $\therefore \quad v(t) = v(0) + \int_0^t a(s)\,ds$

assumption $v(0)=0$ → $= 0 + \int_0^t 3e^{-s}\,ds$.

$= \left[-3e^{-s}\right]_0^t$ ← $(e^{-s})' = -e^{-s}$, $\therefore (-e^{-s})' = e^{-s}$.

$= -3e^{-t} - (-3e^{-0})$

$\therefore \quad v(t) = -3e^{-t} + 3$.

$\therefore \quad x(t) = x(0) + \int_0^t v(s)\,ds = 1 + \int_0^t (-3e^{-s} + 3)\,ds = 1 + \left[3e^{-s} + 3s\right]_0^t$

$x(0) = 1$ assumption.

$= 1 + (3e^{-t} + t) - (3e^{-0} + 0) = -2 + 3e^{-t} + t$ ∎