APPLICATIONS OF THE DEFINITE INTEGRAL (CHAP. 9.5)

**Average Value of a Function f(x)** on the interval \([a, b]\) is

\[
\bar{f} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

1 over the length of the interval.

5. Find the average value of \( f(x) = \sqrt{x} \) on \([0, 4]\).

\[
\bar{f} = \frac{1}{4-0} \int_0^4 \sqrt{x} \, dx = \frac{1}{4} \left( \frac{2}{3} x^{3/2} \right) \bigg|_0^4
\]

\[
= \frac{1}{4} \left( \frac{2}{3} \cdot 4^{3/2} \right) = \frac{8}{6} = \frac{4}{3}
\]
CHAPTER 5: MASS DENSITY & CENTER OF MASS

INTRODUCTION TO PROBLEM:

For N objects distributed along a line, with positions xi & masses mi, the center of mass is:

\( x = \frac{x_1 m_1 + x_2 m_2 + \ldots + x_n m_n}{m_1 + m_2 + \ldots + m_n} \)

\( M = \sum m_i \) total mass

\( x = \frac{1}{M} \sum x_i m_i \)

WHAT DOES THE CENTER OF MASS REPRESENT? It is a balance point. Examples follow.

8a. \( M_1 \) & \( M_2 \)

IF \( M_1 = M_2 \) then \( x = \frac{1}{2} \sum x_i m_i \)

SO THE CENTER OF MASS IS AT 0, that is in the center.
If \( M_1 = 2M_2 \), then \( x = \frac{-x_1M_1 + x_2M_2}{3M_2} \).

\[
x = \frac{-x_1 + x_2}{3} = \frac{-x_1}{3} + \frac{x_2}{3}
\]

Now the center of mass is closer to the heavier object!

Centre of mass, that is the balance point.

Goal for today:

Find the mass & center of mass of a bar whose density (mass per unit length) changes gradually along the bar.

BAR

THINNER MATERIAL

THICKER MATERIAL
Given the density \( \rho(x) \) for \( 0 \leq x \leq L \), where \( L \) is the length of the bar.

**Density** = Mass / Length, so mass = density \( \times \) length.

Split the interval \([0, L]\) into \( N\) small pieces of length \( \Delta x = \frac{L}{N} \).

Using points \( x_0, x_1, \ldots, x_N \).

\( k^{th} \) piece of the bar.

Let \( m_k \) be the mass of the \( k^{th} \) piece. Then

\[
m_k = \rho(x_k) \cdot \Delta x
\]

We approximate the density on \([x_{k-1}, x_k]\) by density at \( x_k \).
The total mass is approx. by

\[ \sum_{k=1}^{N} p(x_k) \Delta x \]

The bigger \( N \) we take, the better the approximation. So the total mass is

\[ M = \lim_{N \to \infty} \sum_{k=1}^{N} p(x_k) \Delta x \]

\[ = \int_{a}^{b} p(x) dx \]

Since the expression above is the Riemann sum for \( p(x) \) on \([a, b]\).

This is a formula for the mass given the density; it should be memorized.
By the same logic, the Centre of Mass is given by:

\[ \mathbf{r}_{cm} = \frac{\sum \mathbf{r}_i \cdot m_i}{\sum m_i} \]

**FORMULA FOR THE CENTER OF MASS GIVEN THE DENSITY.**

\[ \frac{A}{M} \int_0^L x \cdot \rho(x) \, dx \]

**Q:** A bar of length \( L \) has density \( \rho(x) = 4x \) kg/m for \( 0 < x < L \). Find the total mass, the average density of the bar, find the center of mass, where should we cut the bar to get 2 pieces of equal mass?
SOLUTION:

1) \[ M = \int_0^L 4x^2 \, dx = \left[ \frac{4}{3}x^3 \right]_0^L = \frac{4}{3}L^3 \]

2) \[ \bar{P} = \frac{1}{L} \int_0^L 4x \, dx = \frac{1}{L} \left[ 2x^2 \right]_0^L = 2L \]

AVERAGE DENSITY

3) \[ \bar{x} = \frac{1}{2L^2} \int_0^L x \cdot 4x \, dx \]

CENTER OF MASS

\[ \bar{x} = \frac{1}{2L^2} \int_0^L (4 + \frac{4}{3}x^3) \, dx = \frac{2L}{3} \]

BALANCE POINT

4) WITH SAMANTHA!

IMPORTANT TO KNOW: THE ANSWER TO (A) THE CUTPOINT IS NOT THE SAME AS THE CENTER OF MASS.