

M 103: Midterm 2 Review Exercises

1) In the city of Westeros, babies are born at a rate of

$$b(t) = 1000 + 200 \sin\left(\frac{\pi t}{6}\right)$$

per month, where $t \in [0, 12]$ is the number of months since January 1 in any given year. The people of Westeros die at a rate of

$$d(t) = 900 + 100 \cos\left(\frac{\pi t}{6}\right)$$

people per month.

(a) [5 marks] Find the total change in the population of Westeros between January 1 and December 31 of any given year.

(b) If the population of Westeros was 500,000 on January 1, 2000, find

(i) [1 mark] the population on December 31, 2009;

(ii) [3 marks] the population on March 31, 2010.

2) Evaluate the following integrals

(a) $\int \frac{\tan^2(x)}{\sin(x)} dx.$

(b) $\int \frac{1}{x^2 - 6x + 10} dx.$

(c) $\int \sin(2x) \cos(x) dx.$

3) Find the volume of the object obtained by rotating the region bounded by the x -axis, y -axis and the function $f(x) = x^2 - 2x + 1$ about the x -axis.

4) The cumulative function F for a continuous random variable x taking values in $[0, 5]$ satisfies: $F(1) = 1/4$, $F(2) = 1/2$ and $F(3) = 4/5$.

The probability that $(x - 2)^2$ is less than 1 equals

5) Consider the function

$$f(x) = C \sin\left(\pi \frac{x}{2}\right), \quad \text{for } 0 \leq x \leq 1.$$

For what $C > 0$, does $f(x)$ become a probability density? What is the mean value?

6) Use separation of variables to solve the following differential equation with given initial conditions:

$$\frac{dy}{dt} = ty, \quad y(0) = 1.$$

7) Evaluate $\lim_{x \rightarrow 0} \frac{1 + x - e^x}{1 - \cos(3x)}$ using L'Hopital's rule.

8) Evaluate $\int_0^1 (3x^2 + 1) \arctan(x) dx$. (Hint: Use integration by parts.)

9) Find the average value of $f(x) = \frac{\ln(x)}{x}$ between $x = 1$ and $x = e$.

10) Evaluate the following integrals or explain why they diverge.

(a) $\int_0^{\pi/6} \frac{\cos(x)}{\sqrt{\sin(x)}} dx$

(b) $\int_1^{\infty} \frac{dx}{x - e^{-x}}$

11) Write an integral that gives the length of the curve $y = \ln |\cos(x)|$ when $0 \leq x \leq \pi$. (Do not evaluate the integral)

12) The probability density function of an individual's lifetime (in years) is $p(x) = C(1 + (x - 20)^2)$ where $0 \leq x \leq 100$.

(a) Find the constant C .

(b) Find the cumulative function associated with this probability density function.

(c) Find the fraction of the population that has died by age 20.

(d) What is the mean lifetime in this population?