M 103-Lecture March 17, 2016

Contents:

Convergence and Divergence

Our previous notions of convergence and divergence extend to sequences, where a sequence is said to converge if it eventually approaches a fixed value (otherwise, it is divergent). We now provide a rigorous definition of this concept.

Definition: The sequence $(a_k)_{k\geq 0}$ converges to the limit a_∞ as $k \to \infty$ if, for any $\delta > 0$, a number N exists such that for all values of $k > N$, $|a_k - a_\infty| < \delta$ holds. Otherwise, the sequence is divergent.

In words, this says: Given any tolerance $\delta > 0$, the terms in the sequence are eventually at a distance δ from the value a_∞. This concept is illustrated in Figure 1.

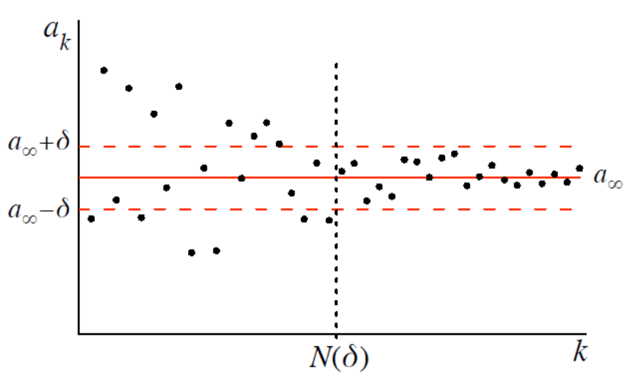

Fig. 1. Figure to illustrate the definition of a convergent sequence. Notice that after the point labeled $N(\delta)$ in the k axis all of the sequence terms are in the interval $[a_\infty - \delta, a_\infty + \delta]$, so that they are at most a distance δ from the limit a_∞.

Notation: If a sequence $(a_k)_{k\geq 0}$ converges to a limit a_∞, we write

$$\lim_{k\to\infty} a_k \to a_\infty,$$

or for brevity

$$a_k \to a_\infty.$$

Let us look at a simple example to consider this definition more in depth

Example 1: Determine if the sequence $\left(\frac{1}{10^k}\right)_{k\geq 0}$ converges or diverges.

Solution: It is apparent that $\frac{1}{10^k} \to 0$. To show this rigorously, consider any tolerance $\delta > 0$. We must find an integer N (which may depend on δ) so that for any $k > N$ we have

$$|a_k - 0| < \delta \Rightarrow 10^{-k} < \delta.$$

This last expression implies $k > \frac{\ln(\delta)}{\ln(10)} = \log_{10}(\delta)$. Therefore we may choose $N = \log_{10}(\delta)$.

In conclusion, given $\delta > 0$, if $k > N := \log_{10}(\delta)$, then $\left|\frac{1}{10^k} - 0\right| < \delta$. So the sequence converges to 0.

Example 2: Determine if the sequence $\left((-1)^k\right)_{k\geq 0}$ converges or diverges.

Solution: Notice that this sequence is given as $(1, -1, 1, -1, 1, -1, ...)$, so that it will be a divergent sequence. To show this rigorously, consider a potential limit a_∞. Then we must have either $a_\infty > 0$ or $a_\infty \leq 0$.

If $a_\infty > 0$, then notice that for k odd we have $(-1)^k = -1$ so that:

$$|a_k - a_\infty| = |(-1)^k - a_\infty| = |-1 - a_\infty| = |1 + a_\infty| = 1 + a_\infty > 1.$$

Therefore, for any $\delta < 1$ we cannot find an integer N as is required. A similar conclusion occurs if we assume $a_\infty \leq 0$ (for even sequence terms in this case), so that no limit exists and the sequence is divergent.

Note: In terms of this definition, the convergence of a sequence does not depend on the "head" of the sequence, since given a tolerance δ and N we only care about the terms after the index $k > N$ (known as the tail of the sequence). In otherwords, changing the first several terms (e.g., million terms) of a sequence will not affect its convergence.

When we can obtain a closed form of a sequence, then we may use this formula and our knowledge of limits to our advantage: If a sequence $(a_k)_{k\geq 0}$ can be given as $a_k = f(k)$, then the sequence converges if and only if the limit $\lim_{x\to\infty} f(x)$ exists (so equally both must converge or diverge). Furthermore, if the limit exists then the sequence converges to this limit, i.e., if $\lim_{x\to\infty} f(x) = L$ then $a_k \to L$.

For example, in the Example 1 above we are given the closed form $(a_k)_{k\geq 0} = \left(\frac{1}{10^k}\right)_{k\geq 0}$ so that in this case we can write $a_k = f(k)$ where $f(x) = \frac{1}{10^x}$. Since $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{10^x} = 0$, we have $a_k = \frac{1}{10^k} \to 0$ as we showed.

Let us look at a more interesting example.

Ex: Determine if the sequence $(\sin(1), 2\sin(1/2), 3\sin(1/3), ...)$ converges or diverges.

Solution: In closed form the sequence is given as $(k\sin(1/k))_{k\geq 1}$, so that we have the expression $a_k = k\sin(1/k)$, so the behaviour of this sequence is equally determined by the limit

$$\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \frac{0}{0}.$$

So this limit is of indeterminate form and we may use L'Hopitals rule to compute

$$\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{\cos(1/x)(1/x)'}{(1/x)'} = \lim_{x\to\infty} \cos(1/x) = \cos(0) = 1.$$

Therefore, we also obtain $a_k = k\sin(1/k) \to 1$.

We now introduce some definitions that are usefull for the study of sequences.

Definition: A sequence $(a_k)_{k\geq 0}$ is bounded if there exists a number $M > 0$ such that $|a_k| \leq M \Rightarrow -M \leq a_k \leq M$ for all $k \geq 0$. Otherwise, the sequence is said to be unbounded.

Note: We can also show a sequence is bounded by showing $M_L \leq a_k \leq M_U$ for all $k \geq 0$, i.e., we have a lower and upper bound which need not be the same.

Examples include the sequences $\left(\frac{(-1)^k}{k}\right)_{k\geq 1}$ and $\left(e^{-k}\right)_{k\geq 0}$. (What are the bounds of these sequences?)

Definition: A sequence $(a_k)_{k\geq 0}$ is monotone or monotonic if either (i) $a_0 \leq a_1 \leq a_2 \leq ...$ or (ii) $a_0 \geq a_1 \geq a_2 \geq ....$ In case (i) we say the sequence is monotonically increasing, and in case (ii) it is monotonically decreasing. Otherwise, the sequence is said to be non-monotone.

Remark: Another way to say that a sequence is increasing is to say that $a_{k+1} \geq a_k$ for all $k \geq 0$. Likewise we can say that a sequence is decreasing if $a_{k+1} \leq a_k$ for all $k \geq 0$.

Examples include the sequences $\left(\frac{1}{\ln(k+2)}\right)_{k\geq 0}$ and $\left(3^k\right)_{k\geq 0}$. (Is each of these examples increasing or decreasing?)

Challenge Question: Give examples of

- a bounded and monotone sequence
- a bounded and non-monotone sequence
- an unbounded and monotone sequence
- an unbounded and non-monotone sequence.

With these definitions, we now make an important observation.

Theorem: A convergent sequence must be bounded.

Indeed, notice that if a sequence a_k converges to a finite value a_∞ the by definition for $\delta = 1$ there exists some N so that for any $k > N$ we have $|a_k - a_\infty| < 1$.

We now consider the head and tail of our sequence in terms of N. Notice that the sequence terms a_k for $k > N$ (the tail), satisfy $|a_k - a_\infty| < 1 \Rightarrow -1 < a_k - a_\infty < 1 \Rightarrow a_\infty - 1 < a_k < a_\infty + 1$, so that $|a_k| < \max\{|a_\infty + 1|, |a_\infty - 1|\}$. On the other hand the sequence terms a_k for $k \leq N$ (the head), are a finite number of terms and therefore bounded by $M = \max\{a_0, a_1, ..., a_N\}$. So all terms are bounded by $\max\{M, |a_\infty + 1|, |a_\infty - 1|\}$.

This observation is nice because we notice that unbounded sequence must be divergent. On the other hand, a bounded sequence need not be convergent. For example the sequence from Example 2 $((-1)^k)_{k\geq 0}$, is bounded (by 1) and was shown to be divergent.

Challenge Question: Give another example of a bounded and divergent sequence.

You may have seen the following Theorem in your previous differential calculus course in terms of limits.

Squeeze Theorem: Suppose that $a_k \leq b_k \leq c_k$ for all $k \geq 0$ and that both sequences $(a_k)_{k\geq 0}, (c_k)_{k\geq 0}$ converge to the same limit $L = a_\infty = c_\infty$. Then the sequence $(b_k)_{k\geq 0}$ must converge to the same limit L.

This result if helpful since we may deal with a seemingly complicated sequence by "squeezing" it between two simpler and easier to handle sequences.

Ex: Determine if the sequence $a_k = \frac{\cos(k)}{k}$ is convergent or divergent.

Solution: Indeed this is a strange sequence, but inuitively as k grows the term $\frac{1}{k}$ goes to zero while the term $\cos(x)$ is bounded. With this intuition we can use the squeeze theorem since $-1 \leq \cos(x) \leq 1$ gives

$$\frac{-1}{x} \leq \frac{\cos(x)}{x} \leq \frac{1}{x}.$$

Since we know $\frac{1}{k} \to 0$ and $\frac{-1}{k} \to 0$, by the squeeze theorem we also have $a_k = \frac{\cos(k)}{k} \to 0$.

We end this section with another theorem that is helpful to determine is a sequence is convergent.

Theorem: A bounded and monotone sequence is convergent.

This result is nice since if we show a sequence is bounded and monotone, then we can conclude it is convergent. Notice that this result does not tell us what the limit is, more work must be done to obtain and actual limit. Let see an example of this theorem in action.

Ex: Consider the sequence given by the recursive formula

$$a_0 = 1, \quad a_{k+1} = a_k + \frac{1}{10^k + 1}.$$

Determine if this sequence converges or diverges.

Solution: To get an idea let us write down a couple of terms of the sequence:

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 1 + \frac{1}{11} \approx 1.0909 \\
a_2 &= 1 + \frac{1}{11} + \frac{1}{101} \approx 1.1008 \\
a_3 &= 1 + \frac{1}{11} + \frac{1}{101} + \frac{1}{1001} \approx 1.1018 \\
a_4 &= 1 + \frac{1}{11} + \frac{1}{101} + \frac{1}{1001} + \frac{1}{10001} \approx 1.1019 \\
&\ldots \\
a_k &= 1 + \sum_{\ell=1}^{k} \frac{1}{10^\ell + 1}
\end{aligned}
$$

From these observations, it seems that the sequence is increasing and bounded above by something. Let us show this.

First, it is easy to see that the sequence is increasing since

$$a_{k+1} = a_k + \frac{1}{10^k + 1} > a_k + 0 \Rightarrow a_{k+1} > a_k,$$

for any k as required. To see that the sequence is bounded, we first easily see that we have 0 as a lower bound since

$$0 < 1 = a_0 \leq a_1 \leq a_2 \leq \ldots$$

where the inequalities follow since we showed the sequence is increasing. For an upper bound, notice from our observation above that for any k

$$
\begin{aligned}
a_k = 1 + \sum_{\ell=1}^{k} \frac{1}{10^\ell + 1} &< 1 + \sum_{\ell=1}^{k} \frac{1}{10^\ell} = \sum_{\ell=0}^{k} \frac{1}{10^\ell} \\
&< \sum_{\ell=0}^{\infty} \frac{1}{10^\ell} = \frac{1}{1 - .1} = \frac{10}{9}.
\end{aligned}
$$

In conclusion, this sequence in monotone (increasing) and bounded $0 < a_k < \frac{10}{9}$ so that by our theorem the sequence must converge, even though we do not know what the limit is!

Iterated Maps

Many sequences we encounter are of the following form: Given a function $g = g(x)$ and a real number v. Evaluating the function $g(x)$ at the value v then yields $g(v)$ some other number, which typically is different from our initial v. Now we can feed the function $g(x)$ this new value $g(v)$ to obtain $g(g(v))$. If we keep doing this the number of brackets involved gets easily out of hands. Because of this we assume in this section the convention of computer scientists and logicians to omit all parenthesis from functions. In other words, we write gv for $g(v)$ and ggv for $g(g(v))$ etc. Following this notation, we can now feed the function $g(x)$ the value ggv and get $gggv$. This process of feeding and re-feeding a function $g = g(x)$ with values v, gv, ggv, ... is called iterating the function $g(x)$ on the value v. Altogether the process yields the sequence

$$(v, gv, ggv, gggv, ...),$$

or recursively

$$a_0 = v, \;\; a_k = g^{[k]}(v).$$

Many sequences are of this form, and we wish to know under what circumstances such sequences converge. In general the behaviour of such sequences depends on the initial point $a_0 = v$ and the function being considered $g(x)$ (we will some of the conditions on g in a bit).

For a continuous function, we have the following observation. Write $g^{[k]}(v)$ as the k-th iterate of $g(x)$ (i.e., applying the function g, k times). Assume the limit $\lim_{k\to\infty} g^{[k]}(v) = L$ exists, then

$$L = \lim_{k\to\infty} g^{[k]}(v) = \lim_{k\to\infty} g(g^{[k-1]}(v)) = g(\lim_{k\to\infty} g^{[k-1]}(v)) = g(L).$$

Therefore, if the limit exists then it must satisfy $g(L) = L$, i.e., it must be a fixed point or also called and equilibrium point.

Note: Notice that this does not mean that if L satisfies $g(L) = L$ then L is the limit. This result only gives us a condition which the limit must satisfy (if it exists).