Lec 24 Ch 9. Differential Equations (D.E.)

- Today: - How to solve simple differential eqns. (§9.1 ~ 9.4.)
  (separation of variables)
  - steady states (§9.3)
- Wed. Translation of physical model into differential equations (§9.5)
  (Container example)
- Friday. Logistic eqn. (§9.6. ~9.8.)

§. Differential eqns. (D.E.)

- Equation for a function $y(t)$ of the form

$$F\left(t, y, \frac{dy}{dt}\right) = 0$$

Some expression involving these

← more generally, $F\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \dots\right) = 0$

e.g. $\frac{dy}{dt} - t = 0 \iff \frac{dy}{dt} = t$

$\frac{dy}{dt} - y = 0 \iff \frac{dy}{dt} = y.$

$\frac{dy}{dx} = x \cdot y.$

$\frac{d^2x}{dt^2} = -g.$ g = constant.

• Solution to a differential equation $F(t, y, \frac{dy}{dt}) = 0$.

Def = a function $y(t)$ satisfying (= solving) the differential eqn.

e.g. $\frac{dy}{dt} = t$ Solution $y(t) = \frac{1}{2}t^2 + C$

$\frac{dy}{dt} = y$ Solution $y(t) = e^t$ ← a particular solution

($y(t) = Ce^t$ is the general solution to this D.E.)

• Initial value problem.

: Find a solution $y(t)$ to a differential eqn.

that satisfies a given initial condition $y(0) = y_0$ ← specific value.

§ A method to solve some simple D.E.

: "Separation of variables"

It works for equations of the form

$\frac{dy}{dt} = f(t)\,g(y)$. e.g. $\frac{dy}{dt} = t^2 \cdot y$

In particular, works for $\frac{dy}{dt} = g(y)$ e.g. $\frac{dy}{dt} = 1 + y^2$.

Method Separate the variables

$$\frac{dy}{dt} = f(t)\,g(y).$$

$$\Rightarrow \quad \frac{dy}{g(y)} = f(t)\,dt$$

only y variable

only t variable

$$\text{So, } \int \frac{dy}{g(y)} = \int f(t)\,dt$$

From this, get expressions involving y & t **without** $\frac{dy}{dt}$.

EX Solve $\frac{dy}{dt} = ty$ with initial condition $y(0)=2$

<sol>. Separation of variables

$$\frac{dy}{y} = t\,dt$$

Integrate both $\int \frac{dy}{y} = \int t\,dt$

get $\ln|y| + C_1 = \frac{t^2}{2} + C_2$

arbitrary constants from indefinite integrals

Therefore, $\ln|y| = \frac{t^2}{2} + C$ ← arbitrary constant

So, $|y| = e^{\left(\frac{t^2}{2}+C\right)}$ $\left(C = C_2 - C_1\right)$

↑ taking exponential

$\therefore |y| = e^{\frac{t^2}{2}+C} = \boxed{e^C} \cdot e^{\frac{t^2}{2}}$

↑ this is an arbitrary constant (>0).

$\therefore y = \boxed{\pm e^C} e^{\frac{t^2}{2}}$

↑ This is ANYWAYS an arbitrary constant (can be ≥ 0 or < 0).

So, can write

$$\underline{y = k \cdot e^{\frac{t^2}{2}}} \qquad k = \text{arbitrary constant.}$$

- Observe. $y(0) = k e^{\frac{0}{2}} = k.$

$\therefore y(t) = y(0) e^{\frac{t^2}{2}}$

↑ initial value of y

- Initial condition: $y(0) = 2$

$\therefore \underline{y(t) = 2e^{\frac{t^2}{2}}}$ ← the final answer. ▨

§ Steady states.

Def A steady state of a D.E.
is the value for which no further change occurs.
Namely, a steady state of a D.E.
is a solution given by a constant function $y(t) = C$

e.g. $y = 0$ is a steady state of $\frac{dy}{dt} = ty$

(since $y(t) = 0 \Rightarrow \frac{dy}{dt} = 0$, $ty = 0$, $\therefore \frac{dy}{dt} = ty$)

e.g $y = 1$ is a steady state of $\frac{dy}{dt} = t(y-1)$.

(since $y(t) = 1 \Rightarrow \frac{dy}{dt} = 0$, $t(y-1) = 0$, $\therefore \frac{dy}{dt} = t(y-1)$.)

-X- Note If the initial value $y(0) = y_0$ of a D.E.
is a steady state,
then the corresponding solution $y(t)$ is constant
$y(t) = y_0$ (← steady state → No change
→ $\frac{dy}{dt} = 0$ → y = constant.

- How to find a steady state of D.E.
: In the D.E. find a constant y_0 where $\frac{dy}{dt} = 0$.
(This was a part of qualitative study of D.E in MATH1102.
We will BRIEFLY cover this below.)

• For D.E. of the form $\frac{dy}{dt} = f(y)$.

only y variable, no t.

e.g.

$\frac{dy}{dt} = f(y)$ ← say, $f(y) = (y-y_0)(y-y_1)(y-y_2)$

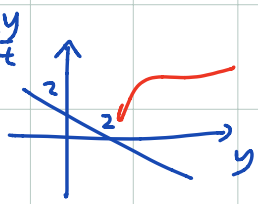
these are where $f(y) = 0$.

and there $\frac{dy}{dt} = 0$

So $y = y_0$, $y = y_1$, $y = y_2$

are constant solutions, i.e. steady states.

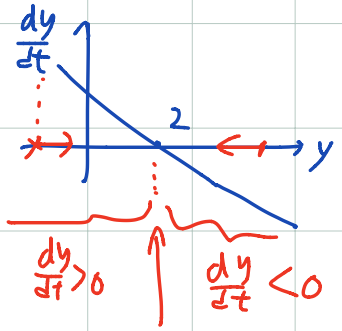
Ex Find steady states of D.E. $\frac{dy}{dt} = 2 - y$

<sol>. 

$\frac{dy}{dt} = 0$ at $y = 2$.

∴ the constant function $y(t) = 2$ is the steady state.

Note No other steady states in this example.



$\frac{dy}{dt} = 0$ at $y = 2$.

Ex D.E. $\frac{dy}{dt} = y(1-y)(2-y)$

(a) Find all steady states.

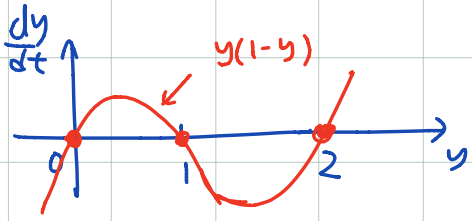
(b) Let $y_1(t)$ be the solution with $y_1(0) = \frac{1}{2}$.

Find $\lim_{t \to \infty} y(t)$

(c) Let $y_2(t)$ be the solution with $y_2(0) = 3$

Find $\lim_{t \to \infty} y_2(t)$.

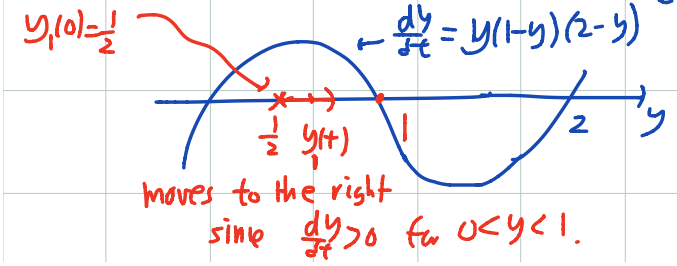
<sol>. sketch the graph $\frac{dy}{dt} = y(1-y)(2-y)$



(a) steady states the values of y where $y(1-y)(2-y) = 0$

$\therefore$ $y=0$, $y=1$, $y=2$ ← all the steady states

(b) Look at the graph of $\frac{dy}{dt} = y(1-y)$.



So. $\lim_{t \to \infty} y_1(t) = 1$

※ · $y_1(t)$ never reaches 1

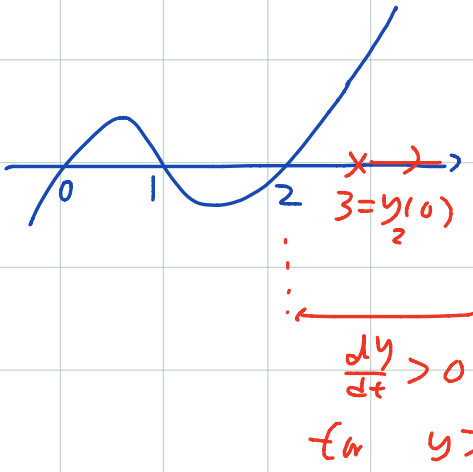
because as it gets closer to 1

$\frac{dy}{dt}$ gets smaller & smaller

decrellerating the movement

$y_1(t)$ gets slower & slower

(c). Look at the graph of $\frac{dy}{dt} = y(1-y)(2-y)$



As time goes on,

∴ $y_2(t)$ becomes larger & large.

· The more value of $y_2(t)$, the larger $\frac{dy}{dt}$.

∴ $y_2(t)$ becomes larger & larger

and the rate of increase gets larger & larger

So, $\lim_{t \to \infty} y_2(t) = +\infty$. ▨

- Below; quantitative method to study D.E:
  - find a solution (explicitly)
  - use the explicit solution to study its behavior

Important example.

EX (Solving D.E. & Finding Steady states).

D.E. $\frac{dy}{dt} = a - by$ & Initial condition $y(0) = y_0$

($a, b \neq 0$ constants)

(a) Find $y(t)$.

(b) Find steady states.

(c) Find $\lim_{t \to \infty} y(t)$ for the case $b > 0$.

<sol>.

(a). • Separation of Variables

$$\frac{dy}{dt} = a - by$$

$$\Rightarrow \frac{dy}{a-by} = dt$$

$$\Rightarrow \underbrace{\int \frac{dy}{a-by}}_{\text{integral in } y\text{-variable}} = \underbrace{\int dt}_{\text{integral in } t\text{-variable}}$$

- $\int dt = t + C_1$

- $\int \frac{dy}{a-by}$

$= \int \frac{1}{u} \cdot \left(\frac{du}{-b}\right)$. ← substitution

$u = a - by$ ← a, b constants

$du = -b\,dy$

$\therefore \frac{du}{-b} = dy$

We use the assumption that $b \neq 0$.

$= -\frac{1}{b} \int \frac{du}{u}$

$= -\frac{1}{b} \ln|u| + C_2$ ← $(\ln|u|)' = \frac{1}{u}$.

- Putting together,

$-\frac{1}{b} \ln|u| + C_2 = t + C_1$

here, $C = C_1 - C_2$ an arbitrary constant!

$-\frac{1}{b} \ln|u| = t + C$

$\ln|u| = -bt + C$ ← arbitrary constant $-b \cdot C$.

- Solve above, by taking exponential.

$|u| = e^{-bt + C}$

- Recall $u = a - by$,

$$\therefore |a - by| = e^{-bt + C}$$

after removing $|\cdot|$ sign we have $+$ or $-$.

$$a - by = \pm e^{-bt + C}$$

$$= \pm e^{C} \cdot e^{-bt}$$

$\pm e^{C}$ ← This is anyways an arbitrary constant, say k.

$$\therefore a - by = k\, e^{-bt}$$ ← k is an arbitrary constant.

$$\therefore a - k e^{-bt} = by$$

$$\therefore y = \frac{1}{b}\left(a - k e^{-bt}\right).$$

- Initial condition: Use $y(0) = y_0$ to determine k.

$$\therefore y(0) = \frac{1}{b}\left(a - k \cdot \underbrace{e^{-b \cdot 0}}_{=1}\right) = y_0$$

$$\Longrightarrow \frac{1}{b}(a - k) = y_0$$

We can determine K:

$$a - K = by_0$$

$$\therefore \quad K = a - by_0$$

- Finally, $y = \frac{1}{b}\left(a - [a-by_0]\, e^{-bt}\right)$

This is a constant. (a) ▨

(b) Steady state: given D.E.

- Can solve $\frac{dy}{dt} = a - by = 0$.

From $a-by=0$, get $y = \frac{a}{b}$. Steady state: $y = \frac{a}{b}$ ▨ (b)

Optional - Can also use the solution (written for a general situation)

$$y = \frac{1}{b}\left(a - [a-by_0]\, e^{-bt}\right)$$

steady state $\Leftrightarrow$ constant solution.

This becomes constant in t for $[a-by_0] = 0$.

i.e. for $y_0 = \frac{a}{b}$.

So, if y starts at $\frac{a}{b}$, then it stays to be constant at $\frac{a}{b}$.

Therefore $y = \frac{a}{b}$ is the (only) steady state.

(c) $\lim_{t\to\infty} y(t)$ (when $b>0$)

$= \lim_{t\to\infty} \frac{1}{b}\left(a - [a-by_0]\, e^{-bt}\right)$ ← the expression for $y(t)$ found from (a).

$= \frac{1}{b}\left(a - [a-by_0]\cdot 0\right)$

$= \frac{a}{b}$.

$\lim_{t\to\infty} e^{-bt}$ for $b>0$: $bt \to \infty$ as $t\to\infty$

$= \lim_{t\to\infty} \frac{1}{e^{bt}}$

$= 0$.

▨ (c).