Last Time: Sequences and the definition of bounded, monotone convergence. Facts: Bounded and monotone then convergent, unbounded implies divergent.

Today: 4 Methods to determine convergence. (b) Iterated maps.

Determining Convergence/Divergence of $(\frac{2n+1}{1-3n})_{n \geq 0}$

1. Take a limit

   Ex: Does $(\left(\frac{2n+1}{1-3n}\right)^2)_{n \geq 0}$ converge or diverges?

   \[
   \lim_{n \to \infty} \left(\frac{2n+1}{1-3n}\right)^2 = \left(\lim_{n \to \infty} \frac{2n+1}{1-3n}\right)^2
   \]

   L'Hopital's Rule

   \[
   = \left(\lim_{n \to \infty} \frac{2}{-3}\right)^2 = \frac{4}{9}
   \]

   We say: $(\left(\frac{2n+1}{1-3n}\right)^2)_{n \geq 0}$ converges to $\frac{4}{9}$.
Tool: comparing growth rates.
Order the following in terms of how fast they grow: $4, x^2, e^x, \frac{1}{x}
\frac{1}{x} \ll 4 \ll x^2 \ll e^x$

[Notation: $f(x) \ll g(x)$, $g(x)$ is much larger than $f(x)$ for large $x$.]

\[
\frac{1}{x} \ll 4 \ll \ln(x) \ll \sqrt{x} \ll x^2 \ll x^3 \ll 2^x \ll e^x \ll 10^x
\]

\[x^p\] constants, lags positive powers, exponentials

Also: $10^0 \ll n! \ll n^n$

Factorial $n! = n \cdot (n-1) \cdot (n-2) \cdots (2) \cdot (1)$
$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
$1! = 1$,
$0! = 1 \leftarrow$ convention.

Ex Determine convergence/divergence
\[
(e^{k(1+k+k^{1000})^{-1}})_{k=0}
\]
\[
\lim_{k \to \infty} e^{k(1+k+k^{1000})} = \lim_{k \to \infty} \frac{e^k}{1+k+k^{1000}} = \infty \quad \text{Numerator grows much faster.}
\]

\[
\left[ \text{i.e. } 1+k+k^{1000} \ll e^k \right]
\]

\[
\left( \frac{k^4 + \ln(k+1)}{\sqrt{5k^7 + 16}} \right)_{k \geq 0}
\]

\[
\lim_{k \to \infty} \frac{k^4 + \ln(k+1)}{\sqrt{5k^7 + 16}} = \lim_{k \to \infty} \frac{k^4}{\sqrt{5k^7}} = \lim_{k \to \infty} \frac{k^4}{\sqrt{5}k^{7/2}} = \lim_{k \to \infty} \frac{k^{7/2}}{\sqrt{5}k^{7/2}} = \infty \quad \text{Diverges}
\]

(2) Squeeze theorem: \( g(x) \leq h(x) \leq f(x) \)

\[
y = 2 \quad \text{and } \lim_{{x \to \infty}} g(x) = \lim_{{x \to \infty}} f(x) = 2
\]

\[
\text{then } \lim_{{n \to \infty}} h(x) = 2.
\]
Ex. \[ \lim_{n \to \infty} \frac{\cos(n)}{n} = 0. \]

\[ \frac{-1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n} \]

\[ \lim_{n \to \infty} -\frac{1}{n} = \lim_{n \to \infty} \frac{-1}{n} = 0 \]

So by squeeze theorem \[ \lim_{n \to \infty} \frac{\cos(n)}{n} = 0. \]

Clicker #2 \[ \lim_{n \to \infty} \frac{5n^2 - \sin(2n)}{n^2 + 10} = 5. \]

\[ \frac{5n^2 - 1}{n^2 + 10} \leq \frac{5n^2 - \sin(2n)}{n^2 + 10} \leq \frac{5n^2 - 1}{n^2 + 10} \]

\[ \lim_{n \to \infty} \frac{5n^2 - 1}{n^2 + 10} = 5 = \lim_{n \to \infty} \frac{5n^2 - 1}{n^2 + 10} \]

by squeeze theorem \[ \lim_{n \to \infty} \frac{5n^2 - \sin(2n)}{n^2 + 10} = 5. \]

3) By Facts: Bounded + Monotone \(\Rightarrow\) convergent
   * unbounded. \(\Rightarrow\) divergent.

4) Facts about iterated maps.
Iterated Maps

Many Forms

- \((x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots)\)

Last time: \(f(x) = 1 + \frac{x}{2}\) and \(x_0 = 1\)
\((1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots)\)

- \((x_0, f(x_0), f(f(x_0), f(f(f(x_0))))\ldots)\)

- \((x_0, f^{[1]}(x_0), f^{[2]}(x_0), f^{[3]}(x_0), \ldots)\)

\[f^{[n]}(x_0) = f(f(\ldots f(x_0) \ldots))\]
\(n\) \(f\) functions.

- \(f^{[1]}(x_0) = f(x_0)\)

- \(f^{[0]}(x_0) = x_0\)

- \((x_0, f^4(x_0), f^2(x_0), f^3(x_0), \ldots)\)

Note: notation can get confusing;
- Recall \(f^{(n)}(x_0)\) means \(n\)th derivative
Most common:

\[ a_{n+1} = f(a_n) \] with initial value \( a_0 = x_0 \)

\begin{align*}
a_0 &= x_0 \\
a_1 &= f(a_0) = f(x_0) \\
a_2 &= f(a_1) = f(f(x_0)) \\
a_3 &= f(a_2) = f(f(f(x_0)))
\end{align*}

Ex: \( a_{n+1} = 1 + \frac{a_n}{2} \), \( a_0 = 1 \)

What is \( f(x) \) where \( a_{n+1} = f(a_n) \)?

\[ f(x) = 1 + \frac{x}{2} \]

\[ \text{From } a_{n+1} = f(a_n) = 1 + \frac{a_n}{2} \]

\( \text{OBBWEBBING (way to visualize)} \)

Sketch \( y = f(x) = 1 + \frac{x}{2} \) and \( y = x \).

\( \text{Sketch } y = f(x) \) and \( y = x \).

Sketch \( y = f(x) \) and \( y = x \).

Sketch \( y = f(x) \) and \( y = x \).
Start at \((a_0, f(a_0))\). Repeat vertically to \((a_0, f(a_0)) = (a_0, a_1)\).

Move horizontally to the line \(y = x\) to \((a_1, a_1)\).

\((a_n)_{n \geq 0} = (1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \ldots)\)

\(a_{n+1} = 1 + \frac{a_n}{2}; \ a_0 = 1\)

\[\lim_{n \to \infty} a_n = 2 \leq \text{matches cobwebbing approaching } (2,2)\]

Ex: Draw the cobweb.

\(a_{n+1} = 4 - 2a_n\) (with \(a_0 = 1\))

Sketch \(y = f(x)\) where \(f(x) = 4 - 2x\)

Sketch \(y = x\)

Start at \((1,1)\) move vertically, to graph \(y = f(x)\).
Colo web doesn't converge to a point, \( \lim_{{n \to \infty}} a_n = \text{DNE.} \)

Why iterated maps?
Recall: often modeling with diff, e.g.
we have \( \frac{dy}{dx} = g(y) \).

Like \( \frac{dy}{dx} = dy \) or \( \frac{dy}{dx} = xy(1-x/k) \)

often modeling with sequences we have \( P_{n+1} = g(P_n) \)

ex. population of ducks doubles every spring from hatching.
pop. after year \( n+1 \) equals twice population out year \( n \).

\( P_{n+1} = 2 \cdot P_n \)

In general: \( P_{n+1} = \alpha \cdot P_n \), initial pop. \( = P_0 \).

Clicker #3 under what conditions will the population grow?
COBWEB VIEW $f(x) = ax$

$\alpha > 1$  $\Rightarrow$  $f(x) = x$

pop grows.

$\alpha < 1$  $\Rightarrow$  $y = x$

pop shrinks to $0$.

Another idea: pop grows $P_{n+1} > P_n$

$\frac{P_{n+1}}{P_n} > 1$

$\alpha > 1$

If $\alpha < 0$ then $P_0$ and $P_1 = -P_0$ (like $a = -1$)

Neg pop. make little sense.

We want to know

1. What initial values give steady state solutions?

2. Do we always converge to steady states?

If so under what conditions?
Answer to (a)

A steady state is a value \( L \) where our sequence becomes:
\[
\underbrace{L, L, L, L, L, L, \ldots}_{\text{All } L}.
\]

So if \( a_{n+1} = f(a_n) \) and initial value \( a_0 \) and \( \{a_n\}_{n=0}^\infty \) then \( f(L) = L \).

**Fact:** All solutions to \( f(x) = x \) are steady states.

Ex: Find \( a_0 \) so \( a_{n+1} = 4 - 2a_n \) (constant).

Solve \( f(x) = x \) where \( f(x) = 4 - 2x \)

\[
4 - 2x = x \\
4 = 3x \\
\frac{4}{3} = x.
\]

Check \( a_0 = \frac{4}{3} \) \( a_1 = 4 - 2a_0 = 4 - 2 \cdot \frac{4}{3} = \frac{12 - 8}{3} = \frac{4}{3} \).