Last Time: Method of Separating Variables

Ch 9 Differential Equations

0) Model with differential eg.
1) Find Steady States / Determine stability / Determine when is inc/Dec
2) Method of sep. var. to solve initial value problems.

Steady States

Ref: An equilibrium/steady state/stable solution is a constant solution to a differential equation.

[i.e. when \( \frac{dy}{dt} = 0 \) for all \( t \)]

Ex of diff eqs:

\[
\frac{dy}{dt} = y(t-2) \quad \frac{dy}{dt} = y + t \quad \frac{dy}{dt} = 1 - y
\]

\[\text{Separable.} \quad \text{Not separable} \quad \text{Autonomous} \]

[i.e. of the form \( \frac{dy}{dt} = f(t)g(y) \)]

[i.e. of the form \( \frac{dy}{dt} = g(y) \)]
\[ \frac{dy}{dt} = y(t-2) = 0 \]
\[ \frac{dy}{dt} = y + t = 0 \]
\[ \frac{dy}{dt} = -y = 0 \]

\[ y = 0 \] steady state solution

\[ y = -t \] Not a steady state solution.

\[ y = 1 \] steady state solution.

\[ \frac{dy}{dt} = 1 - y \] has steady state solution \( y = 1 \)

Graph of \( \frac{dy}{dt} \) vs. \( y \)

\[ g(y) = 1 - y \]

\[ \frac{dy}{dt} > 0 \]
\[ \frac{dy}{dt} < 0 \]

\( y \) is inc. \( y \) is dec.

Phase Diagram.

Plotting solutions
Stable solution looks like \[ y = c \]
Unstable solution looks like \[ c \to \]
Saddle/semi-stable/neither looks like \[ c \to c \to \]

Clicker #1

\[ \frac{dy}{dx} = y(y-1)(y+2) \]

Does this have any stable solutions?

\[ g(y) = y(y-1)(y+2) \]

Ex (Population growth/Decay)

Population will grow at a rate proportional to its size.

\[ \frac{dp}{dt} \]

is proportional to \( P \).

\[ \frac{dp}{dt} = k \cdot P \]

\( \frac{dp}{dt} \) autonomous and separable.
\[
\begin{align*}
\text{solve:} & \quad \frac{dp}{dt} = kp, \quad p(0) = p_0 \\
& \frac{1}{p} \frac{dp}{dt} = k \cdot dt \\
& \int \frac{1}{p} dp = \int k dt \\
& \ln|p| = kt + C_1 \\
& e^{kt + C_1} = p \\
& e^{kt} e^{C_1} = p \\
& C_2 e^{kt} = p \\
& C_2 e^{k \cdot 0} = p_0 \\
& C_2 = p_0 \\
& p(t) = p_0 e^{kt}
\end{align*}
\]

For \( k > 0 \):

- \( \frac{dp}{dt} = kp \)
- \( p' \rightarrow p \)
- \( g(p) = kp \)
- \( p_20 \)
- \( P(t) = p_0 e^{kt} \)
- \( P(t) = p_0 e^{kt} \)

For \( k < 0 \):

- \( g(p) = kp \)
- \( \frac{dp}{dt} = kp \)
- \( \text{stable} \)
- \( P(t) = p_0 e^{kt} \)
Clicker #2

Population Double every year from births

25% mortality rate.

\[ \frac{dp}{dt} = \text{rate of growth} - \text{rate of decline} \]

\[ \frac{dp}{dt} = 2P - 25\% \text{ of } P \]

The differential equation is:

\[ \frac{dp}{dt} = 2P - \frac{1}{4}P \]

\[ = \frac{7}{4}P \]

Ex (Logistic Growth)

Population grows at a rate proportional to its size times 10% from capacity (K).

\[ \frac{dp}{dt} = \alpha \cdot P \cdot \frac{K-P}{K} \]

(assume \( \alpha > 0 \))

\[ \frac{dp}{dt} = \alpha \cdot P \cdot \frac{K-P}{K} = 0 \]

when \( P = 0 \) or \( K \).

\[ K \]

When \( P = 0.5K \),

\[ K \]

unstable. Stable
Find the solution.

\[ \frac{dP}{dt} = \alpha P \left( \frac{k-P}{k} \right) \]

\[ \frac{1}{P} \int \frac{k}{k-P} \, dP = \alpha \int dt \]

\[ \int \frac{k}{P(k-P)} \, dP = \int \alpha \, dt \]

**Partial fractions,**

\[ \frac{k}{P(k-P)} = \frac{A}{P} + \frac{B}{k-P} \]

\[ K = A(k-P) + B \cdot P \]

\[ P=0, \quad K = A \cdot k + 0 \]

\[ 1 = A \]

\[ P=k, \quad K = 0 + B \cdot k \]

\[ 1 = B \]

\[ \int \frac{1}{P} + \frac{1}{k-P} \, dP = \alpha \int dt + C_1 \]

\[ \ln |P| + \ln |k-P| = \alpha t + C_2 \]

\[ \ln \left( \frac{P}{k-P} \right) = \alpha t + C_2 \]
\[ e^{xt} + C_2 = \frac{P}{K-P} \]

\[ C_3 e^{xt} = \frac{P}{K-P} \quad P(0) = P_0 \]

\[ C_3 e^{x^0} = \frac{P_0}{K-P_0} \]

\[ C_3 = \frac{P_0}{K-P_0} \]

\[ \frac{P_0}{K-P_0} e^{xt} = \frac{P}{K-P} \]

\[ \text{algebra} \]

\[ \therefore P(t) = \left( \frac{K}{P_0} \right) e^{xt} + 1 \]

\[ \text{Ex (Newton's Law of Cooling)} \]

\[ \frac{dT}{dt} = \alpha \left( T_s - T \right) \]

\[ \text{(assume } \alpha > 0) \]

\[ \text{rate of temperature change of an object is proportional to the difference between its temperature and the surrounding temperature (Ts)} \]
A cup of tea is 100°C and the room is 20°C. What is the temperature at time t of the cup of tea?

\[ \frac{dT}{dt} = K \left( T_0 - T \right) \]

\[ \frac{1}{T_0 - T} \frac{dT}{dt} = K dt \]

\[ -\ln |T_0 - T| = kt + C_1 \]

\[ \ln |20 - T| = -kt - C_2 \]

\[ C_1 = 20 - 1 \]

\[ C_3 e^{-kt} = 20 - 1 \]

\[ C_3 e^0 = 20 - 100 \]

\[ C_3 = -80 \]
\[ -80 e^{-kt} = 20 - T \]

\[ T = 20 + 80 e^{-kt} \]

Check \( T(0) = 100 \) \( \Rightarrow \) All initial conditions we want.

\[ t \to \infty \Rightarrow T \to 20 \]

\[ \text{Ex.} \quad \text{(Modeling Diseases)} \]

\[ \alpha \cdot I \cdot S \]

Susceptibles.  \( \text{(Not sick now)} \)

Infected.  \( \text{(Sick Now)} \)

\[ \frac{dS}{dt} = -\alpha IS \]

\[ \frac{dI}{dt} = \alpha IS \]

\[ \Rightarrow \quad \frac{dI}{dt} = \alpha I(1-I) \]
\[ g(t) = dI(1-I) \]

\[ \frac{dI}{dt} = aIS - \beta I \]

\[ = dI(1-I) - \beta I \]

\[ = aI - aI^2 - \beta I \]

\[ = I(\alpha - \beta - \alpha I) \]

Zeros \( \alpha + \gamma \), \( \alpha - \beta - \alpha I = 0 \)

\[ \frac{\alpha - \beta}{\alpha} = I \]

\[ I \] is less than 1
\[1 - \frac{\beta}{\alpha} > 0 \quad \Rightarrow \quad \alpha > \beta\]

\[1 - \frac{\beta}{\alpha} < 0 \quad \Rightarrow \quad \alpha < \beta\]

Some part of the population is always sick.

disease dies out.