OSH 2

Math 104 - Section 107

Question 1 *(2 points)* Use the Intermediate Value Theorem (and your calculator) to show that the equation

$$e^x = 5 - x$$

has a solution in the interval $[1, 2]$. Find the solution's first two decimal digits after the dot (you must justify your answer using the Intermediate Value Theorem).

Solution: Write $f(x) = e^x + x$. We need to show that the equation

$$f(x) = 5$$

has a solution in the interval $[1, 2]$ and find its first two decimal digits after the dot.

The function $f(x)$ is everywhere continuous (because it is elementary), so we may apply the IVT to any interval.

- $f(1) = 3.718...$
  $f(2) = 9.389...$
  $f(1) < 5 < f(2)$
  By the IVT, $f(x) = 5$ has a solution in the interval $(1, 2)$.

We now narrow the interval containing the solution until we can be certain of what are the first two decimal digits of the solution.

- $f(1.5) = 5.981...$
  $f(1) < 5 < f(1.5)$
  By the IVT, $f(x) = 5$ has a solution in the interval $(1, 1.5)$.

- $f(1.25) = 4.740...$
  $f(1.25) < 5 < f(1.5)$
  By the IVT, $f(x) = 5$ has a solution in the interval $(1.25, 1.5)$.

- $f(1.37) = 5.305...$
  $f(1.25) < 5 < f(1.37)$
  By the IVT, $f(x) = 5$ has a solution in the interval $(1.25, 1.37)$.

- $f(1.31) = 5.016...$
  $f(1.25) < 5 < f(1.31)$
  By the IVT, $f(x) = 5$ has a solution in the interval $(1.25, 1.31)$.

- $f(1.28) = 4.876...$
  $f(1.28) < 5 < f(1.31)$
  By the IVT, $f(x) = 5$ has a solution in the interval $(1.28, 1.31)$.

- $f(1.29) = 4.922...$
  $f(1.29) < 5 < f(1.31)$
  By the IVT, $f(x) = 5$ has a solution in the interval $(1.29, 1.31)$.

- $f(1.30) = 4.969...$
  $f(1.30) < 5 < f(1.31)$
  By the IVT, $f(x) = 5$ has a solution in the interval $(1.30, 1.31)$.

To conclude, the solution to the equation $e^x = 5 - x$ (or $f(x) = 5$) has the form

$$\boxed{x = 1.30...}$$

Question 2 *(2 points)* Differentiate the following functions:

1. $\frac{x^2}{x^3+1}$

$$\frac{d}{dx}\left(\frac{x^2}{x^3+1}\right) = \frac{(x^2)'(x^3+1)-(x^2)(x^3+1)'}{(x^3+1)^2} = \frac{(2x)(x^3+1)-x^2(3x^2)}{(x^3+1)^2}$$
$$= \frac{2x^4+2x-3x^4}{(x^3+1)^2} = \frac{-x^4+2x}{(x^3+1)^2}$$

2. $x^2 \cdot \ln x \cdot \cos x$

$$\frac{d}{dx}(x^2 \cdot \ln x \cdot \cos x) = (x^2)' \cdot \ln x \cdot \cos x + x^2 \cdot (\ln x)' \cdot \cos x + x^2 \cdot \ln x \cdot (\cos x)'$$
$$= 2x \ln x \, \cos x + x^2 \cdot \frac{1}{x} \cdot \cos x + x^2 \ln x \cdot (-\sin x)$$
$$= 2x \ln x \, \cos x + x \cos x - x^2 \ln x \, \sin x$$

3. $\sqrt[3]{x} + \sqrt{e^x+1}$

$$\frac{d}{dx}(\sqrt[3]{x} + \sqrt{e^x+1}) = \frac{d}{dx}(x^{\frac{1}{3}} + \sqrt{e^x+1})$$
$$= \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{2\sqrt{e^x+1}} \cdot (e^x+1)' = \frac{1}{3}x^{-\frac{2}{3}} + \frac{e^x}{2\sqrt{e^x+1}}$$

4. $\sqrt{\ln(e^x + \sin x)}$.

$$\frac{d}{dx}\sqrt{\ln(e^x+\sin x)} = \frac{1}{2\sqrt{\ln(e^x+\sin x)}} \cdot (\ln(e^x+\sin x))'$$
$$= \frac{1}{2\sqrt{\ln(e^x+\sin x)}} \cdot \frac{1}{e^x+\sin x} \cdot (e^x+\sin x)'$$
$$= \frac{1}{2\sqrt{\ln(e^x+\sin x)}} \cdot \frac{1}{e^x+\sin x} \cdot (e^x+\cos x)$$
$$= \frac{e^x+\cos x}{2(e^x+\sin x)\sqrt{\ln(e^x+\sin x)}}$$

Question 3 *(2 points)* Find the derivative of the the following functions according to the limit definition of the derivative (no credit will be given for other methods).

(a) $f(x) = \frac{1}{x^2+1}$

$$f'(a) = \lim_{x\to a} \frac{\frac{1}{x^2+1} - \frac{1}{a^2+1}}{x-a} = \lim_{x\to a} \frac{\left(\frac{(a^2+1)-(x^2+1)}{(x^2+1)(a^2+1)}\right)}{x-a}$$
$$= \lim_{x\to a} \frac{\left(\frac{a^2-x^2}{(x^2+1)(a^2+1)}\right)}{x-a} = \lim_{x\to a} \frac{\left(\frac{(a-x)(a+x)}{(x^2+1)(a^2+1)}\right)}{-(a-x)}$$
$$= \lim_{x\to a} -\frac{(a+x)}{(x^2+1)(a^2+1)} = -\frac{2a}{(a^2+1)(a^2+1)} = -\frac{2a}{(a^2+1)^2}$$

(b) $f(x) = \sqrt{x^2 + x}$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+h)^2+(x+h)} - \sqrt{x^2+x}}{h} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{(x+h)^2+(x+h)} - \sqrt{x^2+x}\right)\left(\sqrt{(x+h)^2+(x+h)} + \sqrt{x^2+x}\right)}{h\left(\sqrt{(x+h)^2+(x+h)} + \sqrt{x^2+x}\right)} \\
&= \lim_{h\to 0} \frac{\left((x+h)^2+(x+h)\right) - \left(x^2+x\right)}{h\left(\sqrt{(x+h)^2+(x+h)} + \sqrt{x^2+x}\right)} \\
&= \lim_{h\to 0} \frac{x^2+2hx+h^2+x+h-x^2-x}{h\left(\sqrt{(x+h)^2+(x+h)} + \sqrt{x^2+x}\right)} \\
&= \lim_{h\to 0} \frac{h(2x+h+1)}{h\left(\sqrt{(x+h)^2+(x+h)} + \sqrt{x^2+x}\right)} \\
&= \lim_{h\to 0} \frac{(2x+h+1)}{\left(\sqrt{(x+h)^2+(x+h)} + \sqrt{x^2+x}\right)} \\
&= \frac{(2x+0+1)}{\left(\sqrt{(x+0)^2+(x+0)} + \sqrt{x^2+x}\right)} = \frac{2x+1}{\sqrt{x^2+x}+\sqrt{x^2+x}} = \frac{2x+1}{2\sqrt{x^2+x}}
\end{aligned}
$$

Question 4 *(2 points)* Let

$$
f(x) = \begin{cases} x^3 - x & x > -1 \\ 2x+2 & x \le -1 \end{cases}
$$

(a) Prove that f is differentiable at $x = -1$ and find $f'(-1)$. (Hint: Compute the left and right limits of $\frac{f(-1+h)-f(-1)}{h}$ as h approaches 0 separately.)

We have

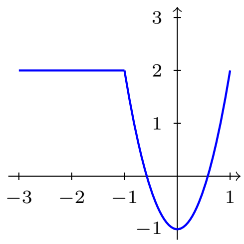
$$
\begin{aligned}
\lim_{h\to 0^+} \frac{f(-1+h)-f(-1)}{h} &= \lim_{h\to 0^+} \frac{\left((-1+h)^3-(-1+h)\right)-(2(-1)+2)}{h} \\
&= \lim_{h\to 0^+} \frac{(-1)^3+3(-1)^2h+3(-1)h^2+h^3+1-h-0}{h} \\
&= \lim_{h\to 0^+} \frac{2h-3h^2+h^3}{h} = \lim_{h\to 0^+} 2-3h+h^2 = 2 \\
\lim_{h\to 0^-} \frac{f(-1+h)-f(-1)}{h} &= \lim_{h\to 0^-} \frac{(2(-1+h)+2)-0}{h} = \lim_{h\to 0^-} \frac{2h}{h} = \lim_{h\to 0^-} 2 = 2
\end{aligned}
$$

Thus, $\lim_{h\to -1} \frac{f(-1+h)-f(-1)}{h}$ exists and equals 2. This means that f is differentiable at $x=-1$ and $f'(-1) = 2$.

(b) Find a formula for $f'(x)$ when $x \neq -1$, and draw the graph of $f'(x)$ on the interval $-3 \le x \le 1$.

When $x > -1$, we have $f(x) = x^3 - x$, so $f'(x) = 3x^2 - 1$.
When $x < -1$, we have $f(x) = 2x + 2$, so $f'(x) = 2$.
Since $f'(-1) = 2$, we get

$$
f'(x) = \begin{cases} 3x^2 - 1 & x > -1 \\ 2 & x \le -1 \end{cases}
$$



Question 5 *(2 points)* Let f and g be differentiable functions.

1. Express the derivatives of the following functions using f, g and their derivatives:

$$x^2 f(x) - g(x), \qquad f(x^2 - g(x)).$$

$$\frac{d}{dx}(x^2 f(x) - g(x)) = (x^2)' f(x) + x^2 f'(x) - g'(x) = 2x f(x) + x^2 f'(x) - g'(x)$$
$$\frac{d}{dx} f(x^2 - g(x)) = f'(x^2 - g(x)) \cdot (x^2 - g(x))' = f'(x^2 - g(x)) \cdot (2x - g'(x))$$

2. It is given that

$$f(2) = 2 \qquad g(2) = 3$$
$$f'(2) = -2 \qquad g'(2) = 1.$$

Find the equation of the tangent line to the graph of $y = \frac{f(x)+1}{g(x)+1}$ at $x = 2$. (Recall: The equation of a line with slope m passing through a point (a, b) is $y = m(x - a) + b$.)

Solution: We have

$$y' = \left(\frac{f(x)+1}{g(x)+1}\right)' = \frac{(f(x)+1)'(g(x)+1) - (f(x)+1)(g(x)+1)'}{(g(x)+1)^2}$$
$$= \frac{f'(x)(g(x)+1) - (f(x)+1)g'(x)}{(g(x)+1)^2}.$$

Thus,
$$y'(2) = \frac{f'(2)(g(2)+1) - (f(2)+1)g'(2)}{(g(2)+1)^2} = \frac{(-2)(3+1) - (2+1)\cdot 1}{(3+1)^2} = -\frac{11}{16}$$

We have $y(2) = \frac{f(2)+1}{g(2)+1} = \frac{2+1}{3+1} = \frac{3}{4}$, so the equation of the tangent line at $x = 2$ is

$$y - \frac{3}{4} = -\frac{11}{16}(x - 2) = -\frac{11}{16}x + \frac{11}{8},$$

or equivalently,

$$\boxed{y = -\frac{11}{16}x + \frac{17}{8}}$$