Question 1  
(2 points) Use the Intermediate Value Theorem (and your calculator) to show that the equation

\[ e^x = 5 - x \]

has a solution in the interval \([1, 2]\). Find the solution’s first two decimal digits after the dot (you must justify your answer using the Intermediate Value Theorem).

Solution: Write \( f(x) = e^x + x \). We need to show that the equation

\[ f(x) = 5 \]

has a solution in the interval \([1, 2]\) and find its first two decimal digits after the dot.

The function \( f(x) \) is everywhere continuous (because it is elementary), so we may apply the IVT to any interval.

- \( f(1) = 3.718... \)
  - \( f(2) = 9.389... \)
  - \( f(1) < 5 < f(2) \)
  - By the IVT, \( f(x) = 5 \) has a solution in the interval \((1, 2)\).

We now narrow the interval containing the solution until we can be certain of what are the first two decimal digits of the solution.

- \( f(1.5) = 5.981... \)
  - \( f(1) < 5 < f(1.5) \)
  - By the IVT, \( f(x) = 5 \) has a solution in the interval \((1, 1.5)\).

- \( f(1.25) = 4.740... \)
  - \( f(1.25) < 5 < f(1.5) \)
  - By the IVT, \( f(x) = 5 \) has a solution in the interval \((1.25, 1.5)\).

- \( f(1.37) = 5.305... \)
  - \( f(1.37) < 5 < f(1.37) \)
  - By the IVT, \( f(x) = 5 \) has a solution in the interval \((1.25, 1.37)\).

- \( f(1.31) = 5.016... \)
  - \( f(1.25) < 5 < f(1.31) \)
  - By the IVT, \( f(x) = 5 \) has a solution in the interval \((1.25, 1.31)\).

- \( f(1.28) = 4.876... \)
  - \( f(1.28) < 5 < f(1.31) \)
  - By the IVT, \( f(x) = 5 \) has a solution in the interval \((1.28, 1.31)\).

- \( f(1.29) = 4.922... \)
  - \( f(1.29) < 5 < f(1.31) \)
  - By the IVT, \( f(x) = 5 \) has a solution in the interval \((1.29, 1.31)\).

- \( f(1.30) = 4.969... \)
  - \( f(1.30) < 5 < f(1.31) \)
  - By the IVT, \( f(x) = 5 \) has a solution in the interval \((1.30, 1.31)\).

To conclude, the solution to the equation \( e^x = 5 - x \) (or \( f(x) = 5 \)) has the form

\[ x = 1.30... \]

Question 2  
(2 points) Differentiate the following functions:
1. \( \frac{x^2}{x^3+1} \)

\[
\frac{d}{dx} \left( \frac{x^2}{x^3+1} \right) = \frac{(x^2)'(x^3+1) - (x^2)(x^3+1)'}{(x^3+1)^2} = \frac{(2x)(x^3+1) - x^2(3x^2)}{(x^3+1)^2} \\
= \frac{2x^4 + 2x - 3x^4}{(x^3+1)^2} = \frac{-x^4 + 2x}{(x^3+1)^2}
\]

2. \( x^2 \cdot \ln x \cdot \cos x \)

\[
\frac{d}{dx} (x^2 \cdot \ln x \cdot \cos x) = (x^2)' \cdot \ln x \cdot \cos x + x^2 \cdot (\ln x)' \cdot \cos x + x^2 \cdot \ln x \cdot (\cos x)' \\
= 2x \ln x \cdot \cos x + x^2 \cdot \frac{1}{x} \cdot \cos x + x^2 \cdot \ln x \cdot (-\sin x) \\
= 2x \ln x \cdot \cos x + x \cos x - x^2 \ln x \cdot \sin x
\]

3. \( \sqrt[3]{x} + \sqrt{e^x + 1} \)

\[
\frac{d}{dx} (\sqrt[3]{x} + \sqrt{e^x + 1}) = \frac{d}{dx} (x^{\frac{1}{3}} + \sqrt{e^x + 1}) \\
= \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{2\sqrt{e^x + 1}} \cdot (e^x + 1)' = \frac{1}{3}x^{-\frac{2}{3}} + \frac{e^x}{2\sqrt{e^x + 1}}
\]

4. \( \sqrt{\ln(e^x + \sin x)} \)

\[
\frac{d}{dx} \sqrt{\ln(e^x + \sin x)} = \frac{1}{2\sqrt{\ln(e^x + \sin x)}} \cdot (\ln(e^x + \sin x))' \\
= \frac{1}{2\sqrt{\ln(e^x + \sin x)}} \cdot \frac{1}{e^x + \sin x} \cdot (e^x + \sin x)' \\
= \frac{1}{2\sqrt{\ln(e^x + \sin x)}} \cdot \frac{1}{e^x + \sin x} \cdot (e^x + \cos x) \\
= \frac{e^x + \cos x}{2(e^x + \sin x)\sqrt{\ln(e^x + \sin x)}}
\]

**Question 3** (2 points) Find the derivative of the following functions according to the limit definition of the derivative (no credit will be given for other methods).

(a) \( f(x) = \frac{1}{x^2+1} \)

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\left(\frac{1}{x^2+1} - \frac{1}{a^2+1}\right)}{x - a} \\
= \lim_{x \to a} \frac{\left(\frac{a^2-x^2}{(x^2+1)(a^2+1)}\right)}{x - a} = \lim_{x \to a} \frac{\left(\frac{(a-x)(a+x)}{(x^2+1)(a^2+1)}\right)}{-a - x} \\
= \lim_{x \to a} \frac{-(a+x)}{(x^2+1)(a^2+1)} = -\frac{2a}{(a^2+1)(a^2+1)} = -\frac{2a}{(a^2+1)^2}
\]
(b) \( f(x) = \sqrt{x^2 + x} \).

\[
f'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)^2 + (x+h) - \sqrt{x^2 + x}}}{h} \\
= \lim_{h \to 0} \frac{(\sqrt{(x+h)^2 + (x+h) - \sqrt{x^2 + x}}) \left( \sqrt{(x+h)^2 + (x+h) + \sqrt{x^2 + x}} \right)}{h \left( \sqrt{(x+h)^2 + (x+h) + \sqrt{x^2 + x}} \right)} \\
= \lim_{h \to 0} \frac{((x+h)^2 + (x+h)) - (x^2 + x)}{h \left( \sqrt{(x+h)^2 + (x+h) + \sqrt{x^2 + x}} \right)} \\
= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + x + h - x^2 - x}{h \left( \sqrt{(x+h)^2 + (x+h) + \sqrt{x^2 + x}} \right)} \\
= \lim_{h \to 0} \frac{h(2x + h + 1)}{h \left( \sqrt{(x+h)^2 + (x+h) + \sqrt{x^2 + x}} \right)} \\
= \lim_{h \to 0} \frac{(2x + h + 1)}{\left( \sqrt{(x+0)^2 + (x+0) + \sqrt{x^2 + x}} \right)} = \frac{2x + 1}{\sqrt{x^2 + x + \sqrt{x^2 + x}}} = \frac{2x + 1}{2\sqrt{x^2 + x}}
\]

**Question 4 (2 points)** Let \( f(x) = \begin{cases} 
  x^3 - x & x > -1 \\
  2x + 2 & x \leq -1 
\end{cases} \)

(a) Prove that \( f \) is differentiable at \( x = -1 \) and find \( f'(-1) \). (Hint: Compute the left and right limits of \( \frac{f(-1+h) - f(-1)}{h} \) as \( h \) approaches 0 separately.)

We have

\[
\lim_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0^+} \frac{((-1+h)^3 - (-1+h)) - (2(-1) + 2)}{h} \\
= \lim_{h \to 0^+} \frac{(-1)^3 + 3(-1)^2 h + 3(-1)h^2 + h^3 + 1 - h - 0}{h} \\
= \lim_{h \to 0^+} \frac{2h - 3h^2 + h^3}{h} = \lim_{h \to 0^+} 2 - 3h + h^2 = 2
\]

\[
\lim_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0^-} \frac{(2(-1+h) + 2) - 0}{h} = \lim_{h \to 0^-} \frac{2h}{h} = \lim_{h \to 0^-} 2 = 2
\]

Thus, \( \lim_{h \to -1} \frac{f(-1+h) - f(-1)}{h} \) exists and equals 2. This means that \( f \) is differentiable at \( x = -1 \) and \( f'(-1) = 2 \).

(b) Find a formula for \( f'(x) \) when \( x \neq -1 \), and draw the graph of \( f'(x) \) on the interval \(-3 \leq x \leq 1\).

When \( x > -1 \), we have \( f(x) = x^3 - x \), so \( f'(x) = 3x^2 - 1 \).

When \( x < -1 \), we have \( f(x) = 2x + 2 \), so \( f'(x) = 2 \).

Since \( f'(-1) = 2 \), we get

\[
f'(x) = \begin{cases} 
  3x^2 - 1 & x > -1 \\
  2 & x \leq -1 
\end{cases}
\]
Question 5 (2 points) Let \( f \) and \( g \) be differentiable functions.

1. Express the derivatives of the following functions using \( f \) and \( g \) and their derivatives:

\[
\frac{d}{dx}(x^2 f(x) - g(x)) = x^2 f'(x) - g'(x)
\]

\[
\frac{d}{dx}(f(x^2) - x^2 g(x)) = 2x f(x) - x^2 g'(x)
\]

2. It is given that

\[
f(2) = 2 \\
g(2) = 3 \\
f'(2) = -2 \\
g'(2) = 1
\]

Find the equation of the tangent line to the graph of \( y = f(x) + 1 \) at \( x = 2 \). (Recall: The equation of a line with slope \( m \) passing through a point \( (a, b) \) is \( y = m(x-a) + b \).)

**Solution:** We have

\[
y' = \left(\frac{f(x) + 1}{g(x) + 1}\right)' = \frac{(f(x) + 1)'(g(x) + 1) - (f(x) + 1)(g(x) + 1)'}{(g(x) + 1)^2} \\
= \frac{f'(x)(g(x) + 1) - (f(x) + 1)g'(x)}{(g(x) + 1)^2}.
\]

Thus,

\[
y'(2) = \frac{f'(2)(g(2) + 1) - (f(2) + 1)g'(2)}{(g(2) + 1)^2} = \frac{(-2)(3 + 1) - (2 + 1) \cdot 1}{(3 + 1)^2} = -\frac{11}{16}
\]

We have \( y(2) = \frac{f(2) + 1}{g(2) + 1} = \frac{2+1}{3+1} = \frac{3}{4} \), so the equation of the tangent line at \( x = 2 \) is

\[
y - \frac{3}{4} = -\frac{11}{16}(x - 2) = -\frac{11}{16}x + \frac{11}{8},
\]

or equivalently,

\[
y = -\frac{11}{16}x + \frac{17}{8}
\]