Question 1  (3.5 points)

The Happy Cookie Bakery is the only producer of cookies in the city of Vancouver. Let $p$ be the price of a cookie (in dollars) and let $q$ be the daily demand for cookies in Genovia. The bakery’s owner estimates that the price and demand are related by the following equation:

$$5000p + 2q = 20000.$$ 

(a) If the bakery sells each cookie for 1$, how many cookies will be sold per day? What will be the factory’s daily revenue? (0.5pt)

We substitute $p = 1$ in $5000p + 2q = 20000$ to get

$$5000 + 2q = 20000$$

$$2q = 15000$$

$$q = 7500$$

Thus, 7500 cookies will be sold.

The daily revenue will be $R = pq = 1 \cdot 7500 = 7500$.

(b) Express the price $p$ in terms of the demand $q$. (0.5pt)

We solve $5000p + 2q = 20000$ for $p$:

$$5000p = 20000 - 2q$$

$$p = \frac{20000 - 2q}{5000} = 4 - \frac{q}{2500}.$$ 

(c) Express the revenue $R$ as a function of $q$. (0.5pt)

$$R = pq = (4 - \frac{q}{2500}) \cdot q = 4q - \frac{q^2}{2500}.$$ 

(d) Suppose that the daily production cost is 5000$ to keep the bakery running plus 0.2$ for every cookie baked. Express the bakery’s daily profit ($P$) as a function of $q$. (0.5pt)

The cost of producing $q$ cookies is $C(q) = 5000 + 0.2q$. Therefore, the profit is:

$$P(q) = R(q) - C(q)$$

$$= (4q - \frac{q^2}{2500}) - (5000 + 0.2q)$$

$$= 4q - \frac{q^2}{2500} - 5000 - 0.2q = -\frac{1}{2500}q^2 + 3.8q - 5000.$$ 

(e) Continuing (d), how many cookies should the bakery produce in order to maximize its profit? What should be the price of a cookie in this case? (0.5pt)

The parabola $P(q) = -\frac{1}{2500}q^2 + 3.8q - 5000$ attains its maximum when

$$q = -\frac{3.8}{2 \cdot (-1/2500)} = \frac{2500}{2} \cdot 3.8 = 4750.$$ 

The factory should produce 4750 cookies in order to maximize its profit. Using part (b), the price of each cookie should then be $4 - \frac{4750}{2500} = 2.18$. 
(f) Assume now that the cost of producing \( q \) cookies each day is \( C(q) = aq + b \) dollars (\( a \) dollars for each cookie and \( b \) dollars to keep the bakery operating). Find \( a \) and \( b \) if the bakery’s daily profit is maximized when \( q = 4000 \) and the maximal daily profit is 5000$. (1pt)

The profit is now given by:

\[
P(q) = R(q) - C(q)
\]

\[
= (4q - \frac{q^2}{2500}) - (aq + b)
\]

\[
= 4q - \frac{q^2}{2500} - aq - b
\]

\[
= -\frac{1}{2500}q^2 + (4 - a)q - b
\]

The parabola \(-\frac{1}{2500}q^2 + (4 - a)q - b\) attains its maximum when \( q = -\frac{4-a}{2 \cdot (-1/2500)} \). It is given the profit is maximal when \( q = 4000 \), hence:

\[
-\frac{4-a}{2 \cdot (-1/2500)} = 4000
\]

\[
-(4-a) = 4000 \cdot 2 \cdot (-\frac{1}{2500})
\]

\[
a - 4 = -3.2
\]

\[
a = 0.8
\]

We also know that the daily profit when \( q = 4000 \) is 5000$, hence:

\[
5000 = P(4000) = -\frac{1}{2500}4000^2 + (4 - a)4000 - b
\]

\[
5000 = -\frac{1}{2500} \cdot 4000^2 + (4 - 0.8)4000 - b
\]

\[
5000 = -6400 + 12800 - b
\]

\[
b = -6400 + 12800 - 5000
\]

\[
b = 1400
\]

**Question 2** (6.5 points) Compute the following limits:

1. \( \lim_{x \to -2} \frac{x^2 + 4x + 4}{x^2 + 3x + 2} \) (0.5pt)

\[
\lim_{x \to -2} \frac{x^2 + 4x + 4}{x^2 + 3x + 2} = \lim_{x \to -2} \frac{(x + 2)^2}{(x + 2)(x + 1)} = \lim_{x \to -2} \frac{(x + 2)}{(x + 1)} = \frac{(-2) + 2}{(-2) + 1} = \frac{0}{-1} = 0
\]

2. \( \lim_{x \to -2} \frac{x^2 + 2x + 2}{x^2 + x + 2} \) (0.5pt)

\[
\lim_{x \to -2} \frac{x^2 + 2x + 2}{x^2 + x + 2} = \frac{(-2)^2 + 2(-2) + 2}{(-2)^2 + (-2) + 2} = \frac{2}{4} = \frac{1}{2}
\]

3. \( \lim_{x \to -2} \frac{x^3 + 8}{x^2 - 4} \) (Hint: \( a^3 + b^3 = (a + b)(a^2 - ab + b^2) \)) (0.5pt)

\[
\lim_{x \to -2} \frac{x^3 + 8}{x^2 - 4} = \lim_{x \to -2} \frac{(x + 2)(x^2 - 2x + 4)}{(x + 2)(x - 2)} = \lim_{x \to -2} \frac{x^2 - 2x + 4}{x - 2} = \frac{(-2)^2 - 2(-2) + 4}{(-2) - 2} = \frac{12}{-4} = -3
\]
4. \(\lim_{x \to 1} \frac{\sqrt{x^2 + 3} - 2}{x - 1}\) 

\[
\lim_{x \to 1} \frac{\sqrt{x^2 + 3} - 2}{x - 1} = \lim_{x \to 1} \frac{\sqrt{x^2 + 3} - 2}{x - 1}(\sqrt{x^2 + 3} + 2) = \lim_{x \to 1} \frac{(x^2 + 3) - 2^2}{(x - 1)(\sqrt{x^2 + 3} + 2)} \\
= \lim_{x \to 1} \frac{x^2 - 1}{(x - 1)(\sqrt{x^2 + 3} + 2)} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{(x - 1)(\sqrt{x^2 + 3} + 2)} \\
= \lim_{x \to 1} \frac{x + 1}{\sqrt{x^2 + 3} + 2} = \frac{1 + 1}{\sqrt{1^2 + 3} + 2} = \frac{2}{4} = \frac{1}{2}
\]

5. \(\lim_{x \to 0} \frac{x(x+2)}{\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1}}\) 

\[
\lim_{x \to 0} \frac{x(x+2)}{\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1}} = \lim_{x \to 0} \frac{x(x+2)(\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1})}{(\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1})(\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1})} \\
= \lim_{x \to 0} \frac{x(x+2)(\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1})}{(x^2 + x + 1) - (x^2 + 1)} \\
= \lim_{x \to 0} \frac{x(x+2)(\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1})}{x} \\
= \lim_{x \to 0} \frac{x + 2}{\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1}} \\
= (0 + 2)(\sqrt{0^2 + 0 + 1} + \sqrt{0^2 + 1}) = 4
\]

6. **Fact:** \(1 + x \leq e^x \leq 1 + x + x^2\) for all \(0 \leq x \leq 1\). 

Use the fact and the squeeze theorem to prove that \(\lim_{x \to 0^+} \frac{e^x - 1}{x} = 1\). 

**Proof:** By the fact, for all \(0 < x \leq 1\), we have 

\[
1 + x \leq e^x \leq 1 + x + x^2 \\
x \leq e^x - 1 \leq x + x^2 \\
\frac{x}{x} \leq \frac{e^x - 1}{x} \leq \frac{x + x^2}{x} \\
1 \leq \frac{e^x - 1}{x} \leq 1 + x
\]

We have 

\[
\lim_{x \to 0^+} 1 = 1 \\
\lim_{x \to 0^+} 1 + x = 1 + 0 = 1
\]

Thus, by the Squeeze Theorem \(\lim_{x \to 0^+} \frac{e^x - 1}{x} = 1\).

7. **Fact:** \(\lim_{x \to 0^+} \frac{\sin x}{x} = 1\). 

Use the fact and limit rules to compute \(\lim_{x \to 0} \frac{x^2 + x^3}{(\sin x)^2}\). 

\[
\lim_{x \to 0} \frac{x^2 + x^3}{(\sin x)^2} = \lim_{x \to 0} \frac{(x + 1)x^2}{(\sin x)^2} \\
= \lim_{x \to 0} (x + 1) \cdot \left(\lim_{x \to 0} \frac{x}{\sin x}\right)^2 \\
= (0 + 1) \cdot \left(\lim_{x \to 0} \frac{1}{\sin x}\right)^2 \\
= 1 \cdot \left(\frac{1}{1}\right)^2 = 1
\]

(We used the Fact in the forth line.)
8. A ball is thrown vertically into the air. It is given that the height of the ball after $t$ seconds is $20t - 5t^2$ meters. 

(a) When will the ball hit the ground? (The ground is height 0.) (0.5pt) **Answer:** We solve

\begin{align*}
20t - 5t^2 &= 0 \\
t(20 - 5) &= 0
\end{align*}

So $t = 0$ or $t = 4$. The ball will hit the ground after **4 seconds**.

(b) Find the average speed of the ball on the interval $1 \leq t \leq 2$. (0.5pt)

$v_{[1,2]} = \frac{(20 \cdot 2 - 5 \cdot 2^2) - (20 \cdot 1 - 5 \cdot 1^2)}{2 - 1} = \frac{20 - 15}{1} = 5$

(c) Compute (according to the definition) the instantaneous speed of the ball at $t = 0$ and $t = 1$. (1pt)

\[
v_0 = \lim_{h \to 0} \frac{(20(0+h) - 5(0+h)^2) - (20 \cdot 0 - 5 \cdot 0^2)}{h} = \lim_{h \to 0} \frac{20h - 5h^2}{h} = \lim_{h \to 0} (20 - 5h) = 20
\]

\[
v_1 = \lim_{h \to 0} \frac{(20(1+h) - 5(1+h)^2) - (20 \cdot 1 - 5 \cdot 1^2)}{h} = \lim_{h \to 0} \frac{(20 + 20h - 5(1 + 2h + h^2)) - (15)}{h} = \lim_{h \to 0} \frac{20 + 20h - 5 - 10h - 5h^2 - 15}{h} = \lim_{h \to 0} \frac{10h - 5h^2}{h} = \lim_{h \to 0} (10 - 5h) = 10 - 5 \cdot 0 = 10
\]