Assignment 1
Math 104 - Section 107

Question 1 *(3.5 points)*

The Happy Cookie Bakery is the only producer of cookies in the city of Vancouver. Let p be the price of a cookie (in dollars) and let q be the daily demand for cookies in Genovia. The bakery's owner estimates that the price and demand are related by the following equation:

$$5000p + 2q = 20000 \ .$$

(a) If the bakery sells each cookie for 1\$, how many cookies will be sold per day? What will be the factory's daily revenue? *(0.5pt)*
We substitute $p = 1$ in $5000p + 2q = 20000$ to get

$$\begin{aligned} 5000 + 2q &= 20000 \\ 2q &= 15000 \\ q &= 7500 \end{aligned}$$

Thus, $\boxed{7500}$ cookies will be sold.
The daily revenue will be $R = p \cdot q = 1 \cdot 7500 = \boxed{7500\$}$.

(b) Express the price p in terms of the demand q. *(0.5pt)*
We solve $5000p + 2q = 20000$ for p:

$$\begin{aligned} 5000p + 2q &= 20000 \\ 5000p &= 20000 - 2q \\ p &= \frac{20000 - 2q}{5000} = \boxed{4 - \frac{q}{2500}} \ . \end{aligned}$$

(c) Express the revenue R as a function of q. *(0.5pt)*
$R = pq = (4 - \frac{q}{2500}) \cdot q = \boxed{4q - \frac{q^2}{2500}}$.

(d) Suppose that the daily production cost is 5000\$ to keep the bakery running plus 0.2\$ for every cookie baked. Express the bakery's daily profit (P) as a function of q. *(0.5pt)*
The cost of producing q cookies is $C(q) = 5000 + 0.2q$. Therefore, the profit is:

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= (4q - \frac{q^2}{2500}) - (5000 + 0.2q) \\ &= 4q - \frac{q^2}{2500} - 5000 - 0.2q = \boxed{-\frac{1}{2500}q^2 + 3.8q - 5000} \ . \end{aligned}$$

(e) Continuing (d), how many cookies should the bakery produce in order to maximize its profit? What should be the price of a cookie in this case? *(0.5pt)*

The parabola $P(q) = -\frac{1}{2500}q^2 + 3.8q - 5000$ attains its maximum when

$$q = -\frac{3.8}{2 \cdot (-1/2500)} = \frac{2500}{2} \cdot 3.8 = 4750 \ .$$

The factory should produce $\boxed{4750}$ cookies in order to maximize its profit. Using part (b), the price of each cookie should then be $4 - \frac{4750}{2500} = \boxed{2.1\$}$.

(f) Assume now that the cost of producing q cookies each day is $C(q) = aq + b$ dollars (a dollars for each cookie and b dollars to keep the bakery operating). Find a and b if the bakery's daily profit is maximized when $q = 4000$ and the maximal daily profit is 5000$. (*1pt*)

The profit is now given by:

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= (4q - \frac{q^2}{2500}) - (aq + b) \\ &= 4q - \frac{q^2}{2500} - aq - b \\ &= -\frac{1}{2500}q^2 + (4-a)q - b \end{aligned}$$

The parabola $-\frac{1}{2500}q^2 + (4-a)q - b$ attains its maximum when $q = -\frac{4-a}{2\cdot(-1/2500)}$. It is given the profit is maximal when $q = 4000$, hence:

$$\begin{aligned} &-\frac{4-a}{2\cdot(-1/2500)} = 4000 \\ &-(4-a) = 4000 \cdot 2 \cdot (-\frac{1}{2500}) \\ &a - 4 = -3.2 \\ &\boxed{a = 0.8} \end{aligned}$$

We also know that the daily profit when $q = 4000$ is 5000$, hence:

$$\begin{aligned} &5000 = P(4000) = -\frac{1}{2500}4000^2 + (4-a)4000 - b \\ &5000 = -\frac{1}{2500} \cdot 4000^2 + (4-0.8)4000 - b \\ &5000 = -6400 + 12800 - b \\ &b = -6400 + 12800 - 5000 \\ &\boxed{b = 1400} \end{aligned}$$

Question 2 *(6.5 points)* Compute the following limits:

1. $\lim\limits_{x\to-2} \frac{x^2+4x+4}{x^2+3x+2}$ (*0.5pt*)

$$\lim_{x\to-2} \frac{x^2+4x+4}{x^2+3x+2} = \lim_{x\to-2} \frac{(x+2)^2}{(x+2)(x+1)} = \lim_{x\to-2} \frac{(x+2)}{(x+1)} = \frac{(-2)+2}{(-2)+1} = \frac{0}{-1} = 0$$

2. $\lim\limits_{x\to-2} \frac{x^2+2x+2}{x^2+x+2}$ (*0.5pt*)

$$\lim_{x\to-2} \frac{x^2+2x+2}{x^2+x+2} = \frac{(-2)^2+2(-2)+2}{(-2)^2+(-2)+2} = \frac{2}{4} = \frac{1}{2}$$

3. $\lim\limits_{x\to-2} \frac{x^3+8}{x^2-4}$ (Hint: $a^3+b^3 = (a+b)(a^2-ab+b^2)$) (*0.5pt*)

$$\lim_{x\to-2} \frac{x^3+8}{x^2-4} = \lim_{x\to-2} \frac{(x+2)(x^2-2x+4)}{(x+2)(x-2)} = \lim_{x\to-2} \frac{(x^2-2x+4)}{(x-2)} = \frac{(-2)^2-2(-2)+4}{(-2)-2} = \frac{12}{-4} = -3$$

4. $\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1}$ *(0.5pt)*

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1} &= \lim_{x\to 1} \frac{(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2)}{(x-1)(\sqrt{x^2+3}+2)} = \lim_{x\to 1} \frac{(x^2+3)-2^2}{(x-1)(\sqrt{x^2+3}+2)} \\
&= \lim_{x\to 1} \frac{x^2-1}{(x-1)(\sqrt{x^2+3}+2)} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(\sqrt{x^2+3}+2)} \\
&= \lim_{x\to 1} \frac{(x+1)}{(\sqrt{x^2+3}+2)} = \frac{1+1}{\sqrt{1^2+3}+2} = \frac{2}{4} = \frac{1}{2}
\end{aligned}$$

5. $\lim_{x\to 0} \frac{x(x+2)}{\sqrt{x^2+x+1}-\sqrt{x^2+1}}$. *(0.5pt)*

$$\begin{aligned}
\lim_{x\to 0} \frac{x(x+2)}{\sqrt{x^2+x+1}-\sqrt{x^2+1}} &= \lim_{x\to 0} \frac{x(x+2)(\sqrt{x^2+x+1}+\sqrt{x^2+1})}{(\sqrt{x^2+x+1}-\sqrt{x^2+1})(\sqrt{x^2+x+1}+\sqrt{x^2+1})} \\
&= \lim_{x\to 0} \frac{x(x+2)(\sqrt{x^2+x+1}+\sqrt{x^2+1})}{(x^2+x+1)-(x^2+1)} \\
&= \lim_{x\to 0} \frac{x(x+2)(\sqrt{x^2+x+1}+\sqrt{x^2+1})}{x} \\
&= \lim_{x\to 0} (x+2)(\sqrt{x^2+x+1}+\sqrt{x^2+1}) \\
&= (0+2)(\sqrt{0^2+0+1}+\sqrt{0^2+1}) = 4
\end{aligned}$$

6. **Fact:** $1+x \le e^x \le 1+x+x^2$ for all $0 \le x \le 1$.
   Use the fact and the squeeze theorem to prove that $\lim_{x\to 0^+} \frac{e^x-1}{x} = 1$. *(1pt)*

   Proof: By the fact, for all $0 < x \le 1$, we have

$$\begin{aligned}
&1+x \le e^x \le 1+x+x^2 && [\text{ subtract } 1 \text{ }] \\
&x \le e^x - 1 \le x + x^2 && [\text{ divide by } x \ (0<x) \text{ }] \\
&\frac{x}{x} \le \frac{e^x-1}{x} \le \frac{x+x^2}{x} \\
&1 \le \frac{e^x-1}{x} \le 1+x
\end{aligned}$$

   We have

$$\begin{aligned}
&\lim_{x\to 0^+} 1 = 1 \\
&\lim_{x\to 0^+} 1+x = 1+0 = 1
\end{aligned}$$

   Thus, by the Squeeze Theorem $\lim_{x\to 0^+} \frac{e^x-1}{x} = 1$.

7. **Fact:** $\lim_{x\to 0} \frac{\sin x}{x} = 1$.
   Use the fact and limit rules to compute $\lim_{x\to 0} \frac{x^2+x^3}{(\sin x)^2}$. *(1pt)*

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2+x^3}{(\sin x)^2} &= \lim_{x\to 0} \frac{(x+1)x^2}{(\sin x)^2} \\
&= \lim_{x\to 0}(x+1) \cdot \left(\lim_{x\to 0} \frac{x}{\sin x}\right)^2 \\
&= (0+1) \cdot \left(\frac{1}{\lim_{x\to 0} \frac{\sin x}{x}}\right)^2 \\
&= 1 \cdot \left(\frac{1}{1}\right)^2 = 1
\end{aligned}$$

   (We used the Fact in the forth line.)

8. A ball is thrown vertically into the air. It is given that the height of the ball after t seconds is $20t - 5t^2$ meters.

(a) When will the ball hit the ground? (The ground is height 0.) (*0.5pt*) **Answer:** We solve

$$\begin{aligned} 20t - 5t^2 &= 0 \\ t(20t - 5) &= 0 \end{aligned}$$

So $t = 0$ or $t = 4$. The ball will hit the ground after $\boxed{\text{4 seconds}}$.

(b) Find the average speed of the ball on the interval $1 \leq t \leq 2$. (*0.5pt*)

$$v_{[1,2]} = \frac{(20 \cdot 2 - 5 \cdot 2^2) - (20 \cdot 1 - 5 \cdot 1^2)}{2 - 1} = \frac{20 - 15}{1} = 5$$

(c) Compute (according to the definition) the instantaneous speed of the ball at $t = 0$ and $t = 1$. (*1pt*)

$$\begin{aligned} v_0 &= \lim_{h \to 0} \frac{(20(0 + h) - 5(0 + h)^2) - (20 \cdot 0 - 5 \cdot 0^2)}{h} = \lim_{h \to 0} \frac{20h - 5h^2}{h} \\ &= \lim_{h \to 0} 20 - 5h = 20 \\ v_1 &= \lim_{h \to 0} \frac{(20(1 + h) - 5(1 + h)^2) - (20 \cdot 1 - 5 \cdot 1^2)}{h} \\ &= \lim_{h \to 0} \frac{(20 + 20h - 5(1 + 2h + h^2)) - (15)}{h} \\ &= \lim_{h \to 0} \frac{20 + 20h - 5 - 10h - 5h^2 - 15}{h} = \lim_{h \to 0} \frac{10h - 5h^2}{h} \\ &= \lim_{h \to 0} 10 - 5h = 10 - 5 \cdot 0 = 10 \end{aligned}$$