OSH 4

Math 104 - Section 107

Question 1 *(3 points)* A framer sells wheat at p dollars per kilogram. The demand q (in kilograms per week) is related to p via via $q^2 + (100p)^2 = 650000$.

1. Find the elasticity when $p = 1\$$.
   Solution: We have $q = \sqrt{650000 - 10000p^2}$, so

$$\epsilon(p) = \frac{p}{q} \cdot \frac{dq}{dq} = \frac{p}{\sqrt{650000 - 10000p^2}} \cdot \frac{-20000p}{2\sqrt{650000^2 - 10000p^2}}$$
$$= \frac{-10000p^2}{650000 - 10000p^2} = \frac{-p^2}{65 - p^2}$$

   Thus,

$$\epsilon(1) = \frac{-1^2}{65 - 1^2} = \frac{-1}{64} = \boxed{-0.015625}$$

2. Assuming $p = 1\$$, estimate the percentage by which the demand will change if p is decreased by 3%.
   Solution:

$$(-3) \cdot \epsilon(1) = \frac{-3}{-64} = 0.046875.$$

   The demand will increase by $\boxed{\frac{3}{64}\% = 0.046875\%}$.

3. Find all values of p for which the demand is elastic (notice that p cannot be arbitrarily large).
   Solution: Observe that $q = \sqrt{650000 - 10000p^2}$ is defined only when $650000 - 10000p^2 > 0$. Solving, we get:

$$\begin{aligned} &10000p^2 < 65000 \\ &p^2 < 65 \\ &p < \sqrt{65} && \text{[since } p > 0\text{]} \end{aligned}$$

   The demand is elastic when $\epsilon(p) < -1$. Using (a), we solve

$$\begin{aligned} &\epsilon(p) < -1 \\ &\frac{-p^2}{65 - p^2} < -1 && \text{[multiply by } 65 - p^2\text{;} \\ &-p^2 < p^2 - 65 && \text{it is positive since } p < \sqrt{65}.\text{]} \\ &65 < 2p^2 \\ &\frac{65}{2} < p^2 \\ &\sqrt{\frac{65}{2}} < p \end{aligned}$$

   The demand is elastic when $\boxed{5.7 \approx \sqrt{\frac{65}{2}} < p < \sqrt{65} \approx 8.06}$.

4. Find the maximal revenue.
   Solution: When the revenue is maximal, $\epsilon(p) = 0$. By the computation in (c), we get that $\epsilon(p) = 0$ when $p = \sqrt{\frac{65}{2}}$. In this case, $q = \sqrt{650000 - 10000p^2} = \sqrt{650000 - \frac{650000}{2}} = \sqrt{\frac{650000}{2}}$. Thus, the maximal revenue is

$$R = pq = \sqrt{\frac{65}{2}}\sqrt{\frac{650000}{2}} = \frac{6500}{2} = \boxed{3250\$}$$

5. The cost function for producing q kilograms of wheat is $C(q) = aq + b$. Find a and b if the maximal profit is $1500_{\$/\text{week}}$ and it is attained when $q = 500_{\text{kg/week}}$.
Solution: We have

$$\begin{aligned} p &= \sqrt{\frac{650000 - q^2}{10000}} = \sqrt{65 - 0.0001q^2} \\ P(q) &= p \cdot q - C(q) = q\sqrt{65 - 0.0001q^2} - aq - b \\ P'(q) &= \sqrt{65 - 0.0001q^2} + q \cdot \frac{-0.0002q}{2\sqrt{65 - 0.0001q^2}} - a \\ &= \sqrt{65 - 0.0001q^2} - \frac{0.0001q^2}{\sqrt{65 - 0.0001q^2}} - a \end{aligned}$$

Since the profit is maximized when $q = 500$, we have $P'(500) = 0$. Thus,

$$\begin{aligned} 0 &= \sqrt{65 - 0.0001 \cdot 500^2} - \frac{0.0001 \cdot 500^2}{\sqrt{65 - 0.0001 \cdot 500^2}} - a \\ 0 &= \sqrt{65 - 25} - \frac{25}{\sqrt{65 - 25}} - a \\ 0 &= \sqrt{40} - \frac{25}{\sqrt{40}} - a \\ a &= \frac{40}{\sqrt{40}} - \frac{25}{\sqrt{40}} = \boxed{\frac{15}{\sqrt{40}} \approx 2.371708} \end{aligned}$$

Since the maximal profit is 1500, we have $P(500) = 1500$. Substituting, we get

$$\begin{aligned} 1500 &= 500\sqrt{65 - 0.0001 \cdot 500^2} - \frac{15}{\sqrt{40}}500 - b \\ 1500 &= 500\sqrt{40} - \frac{7500}{\sqrt{40}} - b \\ b &= -1500 + \frac{20000 - 7500}{\sqrt{40}} = \boxed{\frac{12500}{\sqrt{40}} - 1500 \approx 476.4235376} \end{aligned}$$

Question 2 *(2 points)* In the year 1990, a woman started a saving account with annual interest r (the interest is compounded continuously), and on 2006, the money in the saving account has been doubled.

1. Find r.
Solution: The doubling time is $\frac{\ln 2}{r}$, so $\frac{\ln 2}{r} = 2006 - 1990 = 16$. Thus, $r = \boxed{\frac{\ln 2}{16} \approx 0.0433}$.

2. How many years (from opening the account), did it take for the money in the account to increase by 50%?
Solution: Let C be the initial amount of money in the account. Then after t years, the account has Ce^{rt} dollars. When the money in the account is increased by 50%, we have $Ce^{rt} = 1.5C$. We solve for t:

$$\begin{aligned} Ce^{rt} &= 1.5C \\ e^{rt} &= 1.5 \\ rt &= \ln 1.5 \\ t &= \frac{\ln 1.5}{r} = \boxed{\frac{16 \ln 1.5}{\ln 2} = 16 \log_2 1.5 \approx 9.3594_{\text{years}}} \end{aligned}$$

3. In 2008, the woman deposited additional 2000$ into the account, and in 2014, the account contained 30000$. Find the initial amount of money in the account (i.e. in 1990).
Solution: We write the amount of money in the account in relevant years:

- In 1990: C

- In 2008: $Ce^{18r} + 2000$
- In 2014: $(Ce^{18r} + 2000)e^{6r}$

We solve $(Ce^{18r} + 2000)e^{6r} = 30000$ to find C:

$$
\begin{aligned}
(Ce^{18r} + 2000)e^{6r} &= 30000 \\
Ce^{24r} + 2000e^{6r} &= 30000 \\
Ce^{24r} &= 30000 - 2000e^{6r} \\
C &= \frac{30000 - 2000e^{6r}}{e^{24r}} = 30000e^{-24r} - 2000e^{-18r} \\
C &= \boxed{30000e^{-1.5\ln 2} - 2000e^{-1.125\ln 2} \approx 9689.5977\$}
\end{aligned}
$$

(The answer $30000 \cdot 2^{-1.5} - 2000 \cdot 2^{-1.125}$ and equivalent expressions are also correct.)

4. Find the *rate of change* of the money in the account in the year 2000.
Solution: The rate of change in 2000 is $f'(10)$, where $f(t) = Ce^{rt}$.

$$
\begin{aligned}
f'(t) &= Cre^{rt} \\
f'(10) &= Cre^{r10} = C\frac{\ln 2}{16}(e^{\frac{\ln 2}{16}10}) = C\frac{\ln 2}{16}(e^{\ln 2})^{\frac{10}{16}} = C\frac{\ln 2}{16}2^{\frac{10}{16}} \\
&= \boxed{(30000e^{-1.5\ln 2} - 2000e^{-1.125\ln 2})\frac{\ln 2}{16}2^{\frac{10}{16}} \approx 647.34}
\end{aligned}
$$

Question 3 *(1 point)* The interior of a certain wine cup has the shape of a circular conic (the tip is pointing down, the cone's circular base is parallel to the ground). The cone's radius equals one third of its height. Wine is poured into the glass at a constant rate of 0.5 cubic inches per second. How fast does the wine level in the cup changes when it equals 3 inches?

Variables:

- $h(t)$ — the wine level in the cup (at time t)
- $V(t)$ — the amount of wine in the cup (at time t)

Given: $V'(t) = 0.5_{\text{in}^3/\text{sec}}$.
Relation: The shape of the wine in the cup is a circular cone of hight $h(t)$ and radius $\frac{1}{3}h(t)$. Thus,

$$V(t) = \frac{\pi}{3}\left(\frac{1}{3}h(t)\right)^2 h(t) = \frac{\pi h(t)^3}{27}$$

We differentiate the relation:

$$V'(t) = \frac{\pi}{27} \cdot 3h(t)^2 \cdot h'(t) = \frac{\pi}{9}h(t)^2 h'(t)$$

We substitute $h(t) = 3$ and $V'(t) = 0.5$, to get

$$
\begin{aligned}
0.5 &= \frac{\pi}{9}3^2 h'(t) = \pi h'(t) \\
h'(t) &= \boxed{\frac{1}{2\pi} \approx 0.159_{\text{in/sec}}}
\end{aligned}
$$

Question 4 *(2 points)* Let ABC be a triangle such that $\angle A = 90°$. It is given that AB is increased at a rate of $a_{\text{cm/sec}}$ and AC is increased a rate of $b_{\text{cm/sec}}$.

1. How fast does the area of the triangle changes when $AB = 3_{\text{cm}}$ and $AC = 4_{\text{cm}}$? Express your answer using a and b.
   Variables:
   - $x(t)$ — the length of AB (at time t)
   - $y(t)$ — the length of AC (at time t)
   - $S(t)$ — the area of ABC (at time t)

   Given: $x'(t) = a$ and $y'(t) = b$ (for all t).
   Relation: $S(t) = \frac{x(t)y(t)}{2}$.
   We differentiate the relation:
   $$S'(t) = \frac{1}{2}\left(x'(t)y(t) + x(t)y'(t)\right)$$
   We substitute $x(t) = 3$, $x'(t) = a$, $y(t) = 4$, $y'(t) = b$ to get
   $$S'(t) = \frac{1}{2}(4a + 3b) = \boxed{2a + 1.5b_{\text{cm}^2/\text{sec}}}$$

2. How fast does the length of BC changes when $AB = 3_{\text{cm}}$ and $AC = 4_{\text{cm}}$? Express your answer using a and b.
   Variables: The previous ones and
   - $z(t)$ — the length of BC (at time t)

   Relation: $z(t) = \sqrt{x(t)^2 + y(t)^2}$ (by Pythagoras's Theorem).
   We differentiate the relation (t is omitted for brevity):
   $$z' = \frac{2xx' + 2yy'}{2\sqrt{x^2 + y^2}} = \frac{xx' + yy'}{\sqrt{x^2 + y^2}}$$
   We substitute $x(t) = 3$, $x'(t) = a$, $y(t) = 4$, $y'(t) = b$ to get
   $$z'(t) = \frac{3a + 4b}{\sqrt{3^2 + 4^2}} = \boxed{0.6a + 0.8b_{\text{cm}/\text{sec}}}$$

3. How fast does the angle $\angle B$ changes (in radians per second) when $AB = 3_{\text{cm}}$ and $AC = 4_{\text{cm}}$? Express your answer using a and b.
   Variables: The previous ones and
   - $\theta(t)$ — the angle $\angle B$ in radians (at time t)

   Relation: $\tan\theta(t) = \frac{y(t)}{x(t)}$ (there are other relations such as $\sin\theta(t) = \frac{y(t)}{z(t)} \ldots$).
   We differentiate the relation (t is omitted for brevity):
   $$(1 + (\tan\theta)^2)\theta' = \frac{\theta'}{(\cos\theta)^2} = \frac{y'x - x'y}{x^2}$$
   [Observe that $(\tan\theta)^2 + 1 = \frac{(\sin\theta)^2 + (\cos\theta)^2}{(\cos\theta)^2} = \frac{1}{(\cos\theta)^2}$.]
   We substitute $x(t) = 3$, $x'(t) = a$, $y(t) = 4$, $y'(t) = b$ and $\tan\theta(t) = \frac{y(t)}{x(t)} = \frac{4}{3}$ to get:
   $$\left(1 + \frac{4^2}{3^2}\right)\theta'(t) = \frac{3b - 4a}{3^2}$$
   $$\frac{25}{9}\theta'(t) = \frac{3b - 4a}{9}$$
   $$\theta'(t) = \boxed{\frac{3b - 4a}{25} = 0.12a - 0.16b_{\text{rad}/\text{sec}}}$$

4. Assume that answer to (a) is $5_{\text{cm}^2/\text{sec}}$ and the answer to (b) is $5_{\text{cm/sec}}$. Find a and b.
Solution: We solve

$$\begin{cases} 2a + 1.5b = 5 \\ 0.6a + 0.8b = 5 \end{cases}$$

$$\begin{cases} 6a + 4.5b = 15 \\ 6a + 8b = 50 \end{cases}$$

Subtracting, we get $-3.5b = -35$, so $\boxed{b = 10}$. Substituting in the first equation, we get $2a + 15 = 5$, so $a = (5 - 15)/2 = \boxed{-5}$.

Question 5 *(2 points)* Find the *absolute* minimum and maximum of the function $f(x) = x^3e^x$ on the interval $[-10, 1]$.
Solution: We find critical points by solving

$$\begin{aligned} 0 &= f'(x) = 3x^2e^x + x^3e^x = (3x^2 + x^3)e^x && [\ e^x \neq 0\ ] \\ 0 &= 3x^2 + x^3 = (3 + x)x^2 \\ x &= -3 \text{ or } x = 0 \end{aligned}$$

The derivative $f'(x)$ is defined everywhere, so the only critical points are $x = 0$ and $x = -3$.
We check f on the values $0, -3$ (critical points) and $-10, 1$ (end points of the interval $[-10, 1]$):

$$\begin{aligned} f(-10) &= -1000e^{-10} \approx -0.04539992 \\ f(-3) &= -27e^{-3} \approx -1.3442508 \\ f(0) &= 0e^0 = 0 \\ f(1) &= 1e^1 \approx 2.7182818 \end{aligned}$$

Thus, the absolute maximum of $f(x)$ on $[-10, 1]$ is $\boxed{e \approx 2.718}$ (attained at $x = 1$) and its absolute minimum is $\boxed{-27e^{-3} \approx -1.344}$ (attained at $x = -3$).