The University of British Columbia

November 9th, 2017

Midterm for MATH 104

Closed book examination Time: 50 minutes

Last Name ______________________ **First** ______________

Signature ______________

Student Number ______________

Section Number: __________

Special Instructions:

No memory aids are allowed. No calculators. No communication or other electronic devices. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. **Where boxes are provided for answers, put your final answers in them.**

Midterms written in pencil will not be considered for regrading.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		20
2		10
3		10
4		10
Total		50

1. Short Answer Questions [20 points]. Put your final answer in the box provided, but NO CREDIT will be given for the answer without the correct accompanying work.

(a) [3 points] Find $\frac{dy}{dx}$ when $\cos(xy) = x + y$.

We use implicit differentiation.

$$\frac{d}{dx}\cos(x\,y(x)) = \frac{d}{dx}(x + y(x))$$

$$-\sin(x\,y(x))\left(y(x) + x\frac{dy}{dx}\right) = 1 + \frac{dy}{dx} \quad 1pt$$

$$-\sin(xy)y - 1 = \sin(xy)x\frac{dy}{dx} + \frac{dy}{dx}$$

$$-\sin(xy)y - 1 = \frac{dy}{dx}(1 + \sin(xy)x) \quad 1pt$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sin(xy)y + 1}{\sin(xy)x + 1} \quad 1pt$$

Answer: $\frac{dy}{dx} = -\frac{\sin(xy)y+1}{\sin(xy)x+1}$

(b) [3 points] Find the equation of the normal line to the curve

$$xy + x^2 - 1 = y$$

at the point $(x, y) = (-2, 1)$.

Again we use implicit differentiation

$$y + x\frac{dy}{dx} + 2x = \frac{dy}{dx}$$

$$\frac{dy}{dx}(x-1) = -y - 2x$$

$$m = \frac{dy}{dx} = \frac{-y-2x}{x-1} \quad 1pt$$

slope of the normal line is $-\frac{1}{m} = \frac{x-1}{y+2x}$

at $(x,y) = (-2,1)$: $-\frac{1}{m} = \frac{-2-1}{1-4} = \frac{-3}{-3} = 1$ 1 pt

$\Rightarrow$ equation: $y = 1(x+2) + 1$

$y = x + 3$ 1pt

Answer: $y = x + 3$

(c) [3 points] Let $f(x) = x^{\frac{1}{x}}$ for $x > 0$. Does $f(x)$ have a local maximum? If so, at which value of x?

$f(x) = x^{\frac{1}{x}} = e^{\ln x^{\frac{1}{x}}} = e^{\frac{1}{x}\ln x}$

$f'(x) = e^{\frac{1}{x}\ln x}\left(-\frac{1}{x^2}\ln x + \frac{1}{x^2}\right)$

$= x^{\frac{1}{x}}\left(-\frac{1}{x^2} + \frac{1}{x^2}\right) = x^{\frac{1}{x}-2}(1 - \ln x)$ 1pt

CP's: $\underbrace{x^{\frac{1}{x}-2}}_{\text{never zero, always} > 0}(1 - \ln x) \overset{!}{=} 0 \Leftrightarrow 1 - \ln x = 0$

$1 = \ln x$

$e = x$ 1pt

$\ln x \begin{cases} < 1 & \text{if } x < e \\ > 1 & \text{if } x > e \end{cases} \Rightarrow f'(x)$ changes sign at $x = e$

$\Rightarrow f'(x)$ $+ \searrow -$ $\Rightarrow x = e$ is loc. max 1pt

Answer: $x = e$

(d) [3 points] How much initial investment is required to generate \$1000 in interest over a period of 2 years if the interest rate is 5% per annum, compounded ~~annually~~ continuously? Do not try to simplify your answer.

$C + 1000 = C \cdot e^{0.05 \cdot 2}$ 1pt

~~$\Rightarrow C = 1000e^{-0.1}$~~

$1000 = C(e^{0.1} - 1)$ 1pt

$C = \frac{1000}{e^{0.1} - 1}$ 1pt

Answer: $1000/(e^{0.1} - 1)$

(e) [4 points] If the base b of a triangle is increasing at a rate of 3 centimetres per second, while its height h is decreasing at a rate of 3 centimetres per second, which of the following must be true about the area A of the triangle?

(A) A is always constant.

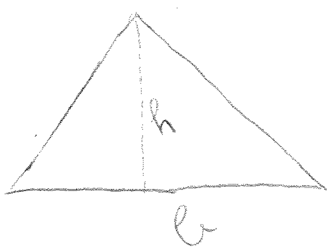
(B) A is always decreasing.

(C) A is always increasing.

(D) A is decreasing only when $b < h$.

(E) A is decreasing only when $b > h$.

Answer: E



$$A = \frac{1}{2} b h$$

$$\frac{dA}{dt} = \frac{1}{2}\left(\underbrace{\frac{db}{dt}}_{3} h + b \underbrace{\frac{dh}{dt}}_{-3}\right) \qquad 2pt$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2}\left(3(h-b)\right)$$

$\Rightarrow$ A is decreasing when $b > h$ 2pt

(f) [4 points] Let $f(x)$ be continuous and differentiable everywhere on the closed interval $[0, 10]$. Suppose that $f(0) = f(10) = 0$ and $f(5) = 4$. Which one of the following is **not** necessarily true:

(A) There is some $c \in (0, 10)$ such that $f(c)$ is a global maximum.

(B) There is some $c \in (0, 10)$ such that $x = c$ is a critical point.

(C) There is some $c \in (0, 5)$ such that $f(c)$ is a local minimum.

(D) There is some $c \in (0, 5)$ such that $f(c) = 2$.

(E) There is some $c \in (0, 5)$ such that $f'(c) = 4/5$.

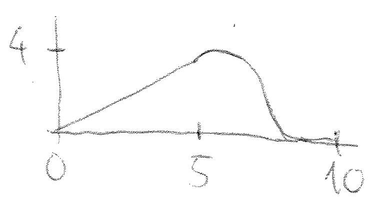
Answer: C

1pt

(A) is always true because of the Extreme Value Theorem: $f(x)$ has to attain its maximum on the interval $[0, 10]$ and this maximum cannot be at the endpoint.

(B) is always true because by part (A) $f(x)$ has a global maximum in $(0, 10)$. This point is a critical point.

(C) is not necessarily true. For example, the function



does not have a local minimum in $(0, 5)$

(D) is always true because of the Intermediate Value Theorem.

(E) is always true because of the Mean Value Theorem

0.5 pt for each

2 [10 points]. The demand curve of a certain product is given by

$$p^2 + 4q + pq = 10,$$

where p is the price in dollars and q is in thousand units. The price elasticity of demand is $\varepsilon(p) = \frac{p}{q}\frac{dq}{dp}$.

(a) [5 points] Compute the price elasticity of demand $\varepsilon(p)$ when the price is $p = \$2$.

$$p^2 + 4q + pq = 10$$
$$q(4+p) = 10 - p^2$$
$$q = \frac{10-p^2}{4+p} \qquad 1\text{pt}$$
$$\frac{dq}{dp} = \frac{-2p(4+p) - (10-p^2)}{(4+p)^2} = \frac{-8p - 2p^2 - 10 + p^2}{(4+p)^2} = \frac{-p^2 - 8p - 10}{(4+p)^2} \qquad 2\text{pt}$$
$$\varepsilon(p) = \frac{p}{q}\cdot\frac{dq}{dp} = p \cdot \frac{(4+p)}{10-p^2}\cdot\frac{(-p^2-8p-10)}{(4+p)^2} = \frac{-p^3 - 8p^2 - 10p}{(4+p)(10-p^2)} \qquad 1\text{pt}$$
$$\varepsilon(2) = \frac{-8-32-20}{6\cdot 6} = -\frac{60}{36} = -\frac{10}{6} = -\frac{5}{3} \qquad 1\text{pt}$$

(b) [2 points] If the price is increased from \$2 by 3%, what is the percentage change in demand?

$$\varepsilon(p) = \%\Delta Q / \%\Delta P$$
$$\Rightarrow \%\Delta Q = \varepsilon(p)\cdot \%\Delta P \qquad 1\text{pt}$$

In this case:

$$\%\Delta Q = -\frac{10}{6}\cdot 0.03 = -0.05 \qquad 1\text{pt}$$

(c) [3 points] Does revenue increase or decrease when the price is increased from $2 by 3%? Justify your answer.

1pt

~~$|\varepsilon(2)| = \frac{2}{3} < 1 \Rightarrow$ price inelastic~~

~~$\Rightarrow$ revenue increases~~

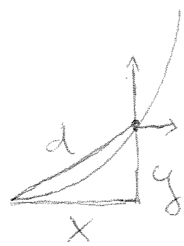
2pt

$|\varepsilon(2)| = \frac{10}{6} > 1 \Rightarrow$ price elastic

$\Rightarrow$ revenue decreases

3 [10 points]. A bug starts at the origin $(x, y) = (0, 0)$ and walks towards the point $(x, y) = (10, 100)$ along the parabola $y = x^2$ in such a way that its distance to the origin increases at a rate of 1 centimetre per minute.

(a) [3 points] At what point on the parabola is the bug's horizontal speed equal to its vertical speed?



$$y = x^2$$

$$y(t) = x^2(t)$$ 1pt

$$\frac{dy}{dt} = \underbrace{2x(t)}_{\text{we need this to be equal to 1}} \frac{dx}{dt}$$ 1pt

$$2x = 1 \Rightarrow \boxed{\begin{array}{l} x = \frac{1}{2} \\ y = x^2 = \frac{1}{4} \end{array}}$$ 1pt

(b) [7 points] Determine the bug's horizontal speed and vertical speed when it is at the point $(x, y) = (2, 4)$.

$$d = \sqrt{x^2 + y^2}$$ 1pt

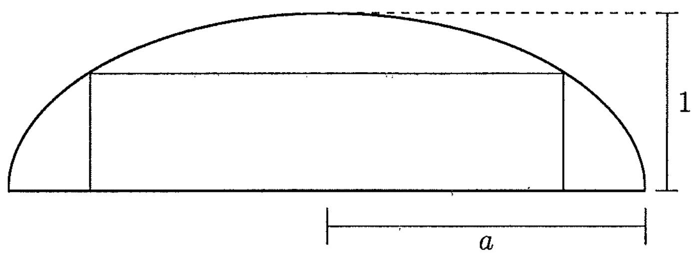
$$d(t) = \sqrt{x^2(t) + y^2(t)}$$ 1pt

$$\underbrace{\frac{dd}{dt}}_{\substack{=1 \\ \text{by the conditions}}} = \frac{1}{2} \frac{1}{\sqrt{x^2+y^2}} \left(2x \frac{dx}{dt} + 2y \underbrace{\frac{dy}{dt}}_{= 2x \frac{dx}{dt} \text{ by part a)}}\right)$$ 1pt 1pt

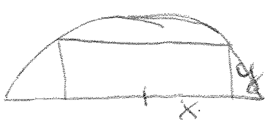
$$\Rightarrow 1 = \frac{1}{2} \frac{1}{\sqrt{x^2+y^2}} (2x + 4xy) \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{2\sqrt{x^2+y^2}}{2x + 4xy}$$ 1pt

at $(x,y) = (2,4)$: $\frac{dx}{dt} = \frac{2\sqrt{4+16}}{4+32} = \frac{2\sqrt{20}}{36} = \frac{\sqrt{5}}{9}$, $\frac{dy}{dt} = 2x\frac{dx}{dt} = \frac{4\sqrt{5}}{9}$ 2pt

4 Let $a > 0$ and consider a half ellipse $\frac{x^2}{a^2} + y^2 = 1$ for $y \geq 0$ with an inscribed rectangle as in the picture below:



(a) [7] Find the area of the largest inscribed rectangle in terms of a.



$A = 2xy$ 1pt $\quad 0 < x < a$

$\frac{x^2}{a^2} + y^2 = 1 \quad \rightarrow \quad y^2 = 1 - \frac{x^2}{a^2}$

$y = \sqrt{1 - \frac{x^2}{a^2}}$ 1pt

$A(x) = 2x\sqrt{1 - \frac{x^2}{a^2}}$

$A'(x) = 2\sqrt{1 - \frac{x^2}{a^2}} + 2x \frac{1}{2} \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}}\left(-\frac{2x}{a^2}\right) \stackrel{!}{=} 0$

$2\sqrt{1 - \frac{x^2}{a^2}} = 2x^2 \frac{1}{a^2} \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}}$ 2pt

$2\left(1 - \frac{x^2}{a^2}\right) = \frac{2x^2}{a^2} \Rightarrow 2 = \frac{4x^2}{a^2} \Rightarrow \frac{a^2}{2} = x^2 \Rightarrow x = \frac{a}{\sqrt{2}}$ 1pt

$A(0) = 0 \quad A(a) = 0$ 1pt

$A\left(\frac{a}{\sqrt{2}}\right) = 2\frac{a}{\sqrt{2}}\sqrt{1 - \frac{\frac{a^2}{2}}{a^2}} = 2\frac{a}{\sqrt{2}\sqrt{2}} = a \rightarrow$ global maximum 1pt

(b) [3] Find the value of a for which the rectangle of largest area is a square.

Rectangle is a square, if $2x = y$ 2pt

$\frac{2a}{\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \underline{\underline{a = \frac{1}{2}}}$ 1pt