Question 1 *(2 points)* Peter Pan sells flying powder. Denote by q the amount of power produced (in grams) and p the price (in dollars) of one gram of power. It is given that:

(i) p and q are related via $p^2 + q^2 = 5000$.

(ii) The cost of producing q grams of powder is $C(q) = 1000 + 10q$.

Answer the following:

1. Find the revenue (R) and profit (P). Express them as functions of q.
   From (i) we get $p^2 = 5000 - q^2$, so $p = \sqrt{5000 - q^2}$. Thus,

$$R(q) = pq = q\sqrt{5000 - q^2}$$
$$P(q) = R(q) - C(q) = q\sqrt{5000 - q^2} - 10q - 1000$$

2. Find the marginal cost and marginal revenue. Express them as functions of q.

$$\begin{aligned} C'(q) &= (1000 + 10q)' = 10 \\ R'(q) &= \sqrt{5000 - q^2} + q \cdot \frac{1}{2\sqrt{5000 - q^2}} \cdot (-2q) \\ &= \sqrt{5000 - q^2} - \frac{q^2}{\sqrt{5000 - q^2}} \end{aligned}$$

3. Suppose $q = 50$. What is the marginal revenue and marginal cost? Does increasing q increases the profit?

$$\begin{aligned} C'(50) &= 10 \\ R'(50) &= \sqrt{5000 - 50^2} - \frac{50^2}{\sqrt{5000 - 50^2}} = 50 - \frac{50^2}{50} = 0 \end{aligned}$$

   We get that $P'(50) = R'(50) - C'(50) = 0 - 10 = -10$. Thus, increasing q decreases the profit.

4. For what q is the profit maximal?
   There was a mistake in this excercise so it will not be graded.

Question 2 *(2 points)* Differentiate the following functions:

1. $2^x + \log_3 x - 2x^\pi$

$$\frac{d}{dx}(2^x + \log_3 x - 2x^\pi) = 2^x \ln 2 + \frac{1}{x \ln 3} - 2\pi x^{\pi - 1}$$

2. $(5^x - x)^{1.4}$

$$\frac{d}{dx}(5^x - x)^{1.4} = 1.4(5^x - x)^{0.4} \cdot (5^x - x)' = 1.4(5^x - x)^{0.4}(5^x \ln 5 - 1)$$

3. $x^{(e^x)}$

$$\begin{aligned} \frac{d}{dx}(x^{(e^x)}) &= \frac{d}{dx}(e^{e^x \ln x}) = e^{e^x \ln x} \cdot (e^x \ln x)' \\ &= x^{(e^x)}\left(e^x \ln x + \frac{e^x}{x}\right) \end{aligned}$$

4. $(\ln x)^{\ln x}$

$$\begin{aligned}\frac{d}{dx}(\ln x)^{\ln x} &= \frac{d}{dx}(e^{\ln x \cdot \ln\ln x}) = e^{\ln x \cdot \ln\ln x} \cdot (\ln x \cdot \ln\ln x)' \\ &= (\ln x)^{\ln x}\left(\frac{\ln\ln x}{x} + \ln x \cdot (\ln\ln x)'\right) \\ &= (\ln x)^{\ln x}\left(\frac{\ln\ln x}{x} + \ln x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}\right) \\ &= (\ln x)^{\ln x}\left(\frac{\ln\ln x}{x} + \frac{1}{x}\right) = \frac{(\ln x)^{\ln x}(\ln\ln x + 1)}{x}\end{aligned}$$

Question 3 *(2 points)* Use implicit differentiation to express $\frac{dy}{dx}$ as a function of x and y in the following cases:

1. $x^3 + xy + y^3 = 1$

$$\begin{aligned}&\frac{d}{dx}(x^3 + xy + y^3) = \frac{d}{dx}1 \\ &3x^2 + y + xy' + 3y^2y' = 0 \\ &y'(x + 3y^2) = -3x^2 - y \\ &\boxed{\frac{dy}{dx} = -\frac{3x^2 + y}{x + 3y^2}}\end{aligned}$$

2. $e^x + e^y = xy + 1$

$$\begin{aligned}&\frac{d}{dx}(e^x + e^y) = \frac{d}{dx}(xy + 1) \\ &e^x + e^y y' = y + xy' \\ &y'(e^y - x) = y - e^x \\ &\boxed{\frac{dy}{dx} = \frac{y - e^x}{e^y - x}}\end{aligned}$$

Question 4 *(2 points)* Find the tangent line to the curve $x + \cos x = y^5 + y^4 - 1$ at the point $(0, 1)$.
Solution: We find $\frac{dy}{dx}$ using implicit differentiation:

$$\begin{aligned}&\frac{d}{dx}(x + \cos x) = \frac{d}{dx}(y^5 + y^4 - 1) \\ &1 - \sin x = 5y^4y' + 4y^3y' \\ &y'(5y^4 + 4y^3) = 1 - \sin x \\ &\frac{dy}{dx} = \frac{1 - \sin x}{5y^4 + 4y^3}\end{aligned}$$

The slope of the tangent line at $(0, 1)$ is

$$m = \left.\frac{dy}{dx}\right|_{x=0,y=1} = \frac{1 - \sin 0}{5 \cdot 1^4 + 4 \cdot 1^3} = \frac{1}{9}\ .$$

Thus, the tangent line is

$$\begin{aligned}&y - 1 = m(x - 0) \\ &\boxed{y = \frac{1}{9}x + 1}\end{aligned}$$

Question 5 *(2 points)* Find all values of a for which the tangent line to the curve $x^2 - axy + y^2 = 1$ at the point $(1, 0)$ passes through the point $(2, 5)$.

Solution: We find $\frac{dy}{dx}$ using implicit differentiation:

$$\frac{d}{dx}(x^2 - axy + y^2) = \frac{d}{dx}(1)$$
$$2x - (ay + axy') + 2yy' = 0$$
$$y'(2y - ax) = ay - 2x$$
$$\frac{dy}{dx} = \frac{ay - 2x}{2y - ax}$$

The slope of the tangent line at $(1, 0)$ is

$$m = \frac{dy}{dx}|_{x=1,y=0} = \frac{a \cdot 0 - 2 \cdot 1}{2 \cdot 0 - a \cdot 1} = \frac{2}{a}$$

The tangent line at $(1, 0)$ is

$$y - 0 = m(x - 1)$$
$$y = \frac{2}{a}(x - 1)$$

The tangent line passes through $(2, 5)$ precisely when

$$5 = \frac{2}{a}(2 - 1)$$
$$5a = 2$$
$$\boxed{a = \frac{2}{5} = 0.4}$$