[15] 1. **Short Answer Questions**. Each question is worth 3 points. Put your final answer in the box provided, but NO CREDIT will be given for the answer without the correct accompanying work.

(a) Let $f(x) = x^3 - 4x + 1$ and let $g(0) = 6$ and let $g'(0) = 6$. Use the quotient rule to find $\left(\frac{f}{g}\right)'(0)$.

$f(0) = 1$

$f'(x) = 3x^2 - 4 \Rightarrow f'(0) = -4$ 1pt

Answer: $-\frac{5}{6}$

$$\left(\frac{f}{g}\right)'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{g^2(0)}$$ 1pt

$$= \frac{-4\cdot 6 - 6}{36} = \frac{-5\cdot 6}{6\cdot 6}$$ 1pt

(b) Evaluate the following limit, be sure to justify your answer.

$$\lim_{x\to\infty} \frac{\sqrt{16x^2 - x + 1}}{5x - 1}$$

$$= \lim \frac{\sqrt{(4x)^2 - x + 1}}{5x-1}$$

Answer: $\frac{4}{5}$

$$= \lim \frac{\sqrt{(4x)^2\left(1 - \frac{1}{4x} + \frac{1}{4x^2}\right)}}{5x\left(1 - \frac{1}{5x}\right)}$$ 1pt

$$= \lim \frac{4x\sqrt{\left(1 - \frac{1}{4x} + \frac{1}{4x^2}\right)}}{5x\left(1 - \frac{1}{5x}\right)} = \frac{4}{5}$$ 1pt 1pt

(c) For what value of c is the following function continuous?

$$f(x) = \begin{cases} x^2 - 5x \, , x \leq 0 \\ \ln(x-c) \, , x > 0 \, . \end{cases}$$

Need:

$0 = \lim_{x \to 0+} \ln(x-c) = \ln(-c)$ 1pt

Apply e^x on both sides: 1pt

$e^0 = -c \Rightarrow c = -1$ 1pt

Answer: $c = -1$

(d) Suppose that $f + g$ is differentiable at 0, and g is not differentiable at 0. Is it true that f is differentiable at 0? Justify your answer for credit.

Let $g(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

Let $f(x) = -g(x)$

Then $(f+g)(x) = 0 \Rightarrow f+g$ is differentiable but f, g are not

Answer: False

(e) Suppose f is continuous and increasing on $[0, 1]$. Is it true that there exists $0 < c < 1$ such that $f(c) = \frac{f(0)+f(1)}{2}$? Justify your answer for credit.

f is increasing on $[0,1]$

$\Rightarrow f(0) < \frac{f(0)+f(1)}{2} < f(1)$

$\Rightarrow$ By the IVT there exists $0 < c < 1$ such that $f(c) = \frac{f(0)+f(1)}{2}$

Answer: True

[10] **2. Definition of the Derivative.**

(a) [3] Carefully state the definition of the derivative of a function $f(x)$ at a point $x = a$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

3pt

or $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(b) [7] Using left and right handed limits, show that the function $f(x) = |3x - 9|$ is not differntiable at $x = 3$.

$$\lim_{x \to 3+} \frac{|3x-9| - |0|}{x-3} \overset{1pt}{=} \lim_{x \to 3+} \frac{3x-9-0}{x-3} \overset{1pt}{=} \lim_{x \to 3+} \frac{3(\cancel{x-3})}{\cancel{x-3}} = 3 \quad 1pt$$

$\neq$

$$\lim_{x \to 3-} \frac{|3x-9| - |0|}{x-3} \overset{1pt}{=} \lim_{x \to 3-} \frac{-3x+9-0}{x-3} \overset{1pt}{=} \lim_{x \to 3-} \frac{-3(x-3)}{x-3} = -3 \quad 1pt$$

$\Rightarrow$ The limit $\lim_{x \to 3} \frac{|3x-9| - |0|}{x-3}$ does not exist

$\Rightarrow$ $f(x) = |3x-9|$ is not differentiable at $x = 3$ 1pt

[10] **3.** When DM sells light aircraft on the open market, they find that when the price is \$4000 per glider they sell 6 gliders per month. For every \$100 in price, the number of gliders they sell decreases by 1. The companies production costs are \$4300 dollars per aircraft, but DM receives a subsidy from the Province of B.C. for \$7000 month.

(a) [2] Find the linear demand equation. Use p for the unit price and q for the monthly demand.

$$q = ap + b$$

$$q = -\frac{1}{100}p + b \qquad 1pt$$

$$6 = -\frac{1}{100}4000 + b \Rightarrow b = 6 + \frac{4000}{100} = 46$$

$$\Rightarrow q = -\frac{1}{100}p + 46 \qquad 1pt$$

$$100q = -p + 4600$$

$$\Rightarrow p = -100q + 4600$$

(b) [1] Find the monthly cost function $C(q)$ *as a function of* q.

$$C(q) = -7000 + 4300q$$

(c) [1] Find the monthly revenue function $R(q)$ *as a function of* q.

$$R(q) = p \cdot q = (-100q + 4600)q$$

$$= -100q^2 + 4600q$$

(d) [3] Find the break-even points. Give both price p and quantity q at each of these points. (You will need the fact that $17^2 = 289$.)

Break-even points: $C(q) = R(q)$ 1pt

$$-7000 + 4300q = -100q^2 + 4600q$$

$$100q^2 - 300q - 7000 = 0$$

$$q^2 - 3q - 70 = 0 \quad \text{1pt}$$

$$q_{1,2} = \frac{3 \pm \sqrt{9 + 280}}{2} = \frac{3 \pm \sqrt{289}}{2} = \frac{3 \pm 17}{2} \begin{cases} 10 & \rightarrow p = 3600 \\ -7 & \rightarrow p = 5300 \end{cases} \quad \text{1pt}$$

(e) [2] Find the derivative of the profit function $P'(q)$. This is often called the *Marginal Profit.*

$$P(q) = R(q) - C(q) = -100q^2 + 4600q - 4300q + 7000$$
$$= -100q^2 + 300q + 7000$$

$$P'(q) = -200q + 300$$

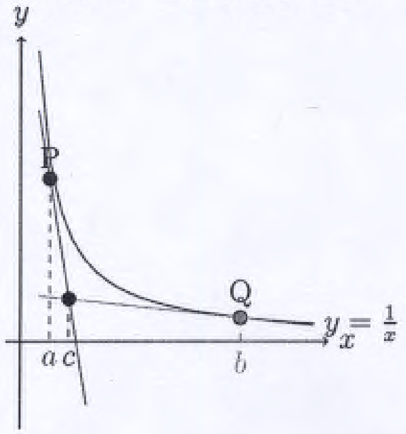
(f) [1] Find the quantity q for which the Marginal Profit is 0.

$$0 = -200q + 300$$

$$200q = 300$$

$$q = \frac{300}{200} = \frac{3}{2}$$

[15] 4. Let $f(x) = \frac{1}{x}$ and c be the x-coordinate of the intersection of the tangent lines going through $P = (a, f(a))$ and $Q = (b, f(b))$ as shown in the diagram.



(a) [5] What is the equation of the tangent line to $y = f(x)$ at the point Q?

$$\ell_Q(x) = f'(b)(x-b) + f(b) \qquad f'(x) = -\frac{1}{x^2}$$

$$\Rightarrow \ell_Q(x) = -\frac{1}{b^2}(x-b) + \frac{1}{b}$$

(b) [5] When $a = 3$ and $b = 5$, calculate c.

$$\left.\begin{array}{l}(1)\ \ell_P(x) = -\frac{1}{3^2}(x-3) + \frac{1}{3} \\ (2)\ \ell_Q(x) = -\frac{1}{5^2}(x-5) + \frac{1}{5}\end{array}\right\} \quad -\frac{1}{9}x + \frac{2}{3} \stackrel{!}{=} -\frac{1}{25}x + \frac{2}{5}$$

$$-x + 6 = -\frac{9}{25}x + \frac{18}{5}$$

$$-25x + 150 = -9x + 90$$

$$60 = 16x \Rightarrow c = x = \frac{60}{16} = \frac{15}{4}$$

(c) [5] Show that for any choice of a and b, c is always given by the formula $c = \frac{2ab}{a+b}$ (c is the harmonic mean of a and b).

$$-\frac{1}{a^2}(x-a) + \frac{1}{a} \stackrel{!}{=} -\frac{1}{b^2}(x-b) + \frac{1}{b}$$

$$-b^2(x-a) + ab^2 = -a^2(x-b) + a^2b$$

$$-b^2x + b^2a + ab^2 = -a^2x + a^2b + a^2b$$

$$x(a^2 - b^2) = 2ab(a-b)$$

$$x = \frac{2ab(a-b)}{a^2-b^2} = \frac{2ab(a-b)}{(a+b)(a-b)}$$

$$\Rightarrow c = x = \frac{2ab}{a+b}$$