[15] 1. **Short Answer Questions**. Each question is worth 3 points. Put your final answer in the box provided, but NO CREDIT will be given for the answer without the correct accompanying work.

(a) Let $f(x) = x^3 - 4x + 1$ and let $g(0) = 2$ and let $g'(0) = 5$. Use the quotient rule to find $\left(\frac{f}{g}\right)'(0)$.

$f(0) = 1$

$f'(x) = 3x^2 - 4 \Rightarrow f'(0) = -4$ 1pt

Answer: $-\frac{13}{4}$

$\left(\frac{f}{g}\right)'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{g^2(0)} =$ 1pt

$= \frac{-4 \cdot 2 - 5}{4} = -\frac{13}{4}$ 1pt

(b) Evaluate the following limit, be sure to justify your answer.

$$\lim_{x\to\infty} \frac{\sqrt{9x^2 - x + 1}}{4x - 5}$$

$= \lim_{x\to\infty} \frac{\sqrt{(3x)^2 - x + 1}}{4x - 5}$

Answer: $\frac{3}{4}$

$= \lim_{x\to\infty} \frac{\sqrt{(3x)^2\left(1 - \frac{1}{3x} + \frac{1}{3x^2}\right)}}{4x\left(1 - \frac{5}{4x}\right)}$ 1pt

$= \lim_{x\to\infty} \frac{3x\sqrt{1 - \frac{1}{3x} + \frac{1}{3x^2}}}{4x\left(1 - \frac{5}{4x}\right)} = \frac{3}{4}$ 1pt 1pt

(c) For what value of c is the following function continuous?

$$f(x) = \begin{cases} x^2 - 3x, x \leq 0 \\ \ln(x-c), x > 0. \end{cases}$$

Need:

$0 = \lim_{x \to 0^+} \ln(x-c) = \ln(-c)$ 1pt

Apply e^x on both sides: 1pt

$e^0 = -c \Rightarrow c = -1$ 1pt

Answer: $c = -1$

(d) Suppose f is continuous and increasing on $[0, 1]$. Is it true that there exists $0 < c < 1$ such that $f(c) = \frac{f(0)+f(1)}{2}$? Justify your answer for credit.

f is increasing on $[0,1]$

$\Rightarrow f(0) < \frac{f(0)+f(1)}{2} < f(1)$

$\Rightarrow$ By the IVT there exists $0 < c < 1$ such that $f(c) = \frac{f(0)+f(1)}{2}$

Answer: True

(e) Suppose that $f \cdot g$ is differentiable at 0, and g is not differentiable at 0. Is it true that f is differentiable at 0? Justify your answer for credit.

Let $f(x) = g(x) = |x|$

Then $f \cdot g = x^2$ is differentiable at 0, but f and g are not.

Answer: False

[10] **2. Definition of the Derivative.**

(a) [3] Carefully state the definition of the derivative of a function $f(x)$ at a point $x = a$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

3pt

or $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(b) [7] Using left and right handed limits, show that the function $f(x) = |2x - 6|$ is not differntiable at $x = 3$.

$$\lim_{x \to 3^+} \frac{|2x-6| - |0|}{x-3} \overset{1pt}{=} \lim_{x \to 3^+} \frac{2x-6-0}{x-3} \overset{1pt}{=} \lim_{x \to 3^+} \frac{2\cancel{(x-3)}}{\cancel{x-3}} = 2 \quad 1pt$$

$\neq$

$$\lim_{x \to 3^-} \frac{|2x-6| - |0|}{x-3} \underset{1pt}{=} \lim_{x \to 3^-} \frac{-2x+6-0}{x-3} \underset{1pt}{=} \lim_{x \to 3^-} \frac{-2\cancel{(x-3)}}{\cancel{x-3}} = -2 \quad 1pt$$

$\Rightarrow$ The limit $\lim_{x \to 3} \frac{|2x-6| - |0|}{x-3}$ does not exist

$\Rightarrow$ $f(x) = |2x-6|$ is not differentiable at $x = 3$ 1pt

[10] **3**. When DM sells light aircraft on the open market, they find that when the price is \$4000 per glider they sell 6 gliders per month. For every \$100 in price, the number of gliders they sell decreases by 1. The companies production costs are \$4300 dollars per aircraft, but DM receives a subsidy from the Province of B.C. for \$7000 month.

(a) [2] Find the linear demand equation. Use p for the unit price and q for the monthly demand.

$$q = ap + b$$

$$q = -\frac{1}{100}p + b \qquad 1pt$$

$$6 = -\frac{1}{100}4000 + b \Rightarrow b = 6 + \frac{4000}{100} = 46$$

$$\Rightarrow q = -\frac{1}{100}p + 46 \qquad 1pt$$

$$100q = -p + 4600$$

$$\Rightarrow p = -100q + 4600$$

(b) [1] Find the monthly cost function $C(q)$ *as a function of* q.

$$C(q) = -7000 + 4300q$$

(c) [1] Find the monthly revenue function $R(q)$ *as a function of* q.

$$R(q) = p \cdot q = (-100q + 4600)q$$

$$= -100q^2 + 4600q$$

(d) [3] Find the break-even points. Give both price p and quantity q at each of these points. (You will need the fact that $17^2 = 289$.)

Break-even points: $C(q) = R(q)$ 1pt

$$-7000 + 4300q = -100q^2 + 4600q$$

$$100q^2 - 300q - 7000 = 0$$

$$q^2 - 3q - 70 = 0 \quad \text{1pt}$$

$$q_{1,2} = \frac{3 \pm \sqrt{9+280}}{2} = \frac{3 \pm \sqrt{289}}{2} = \frac{3 \pm 17}{2} \begin{cases} 10 & \rightarrow p = 3600 \\ -7 & \rightarrow p = 5300 \end{cases}$$

1pt

(e) [2] Find the derivative of the profit function $P'(q)$. This is often called the *Marginal Profit.*

$$P(q) = R(q) - C(q) = -100q^2 + 4600q - 4300q + 7000$$

$$= -100q^2 + 300q + 7000$$

$$P'(q) = -200q + 300$$

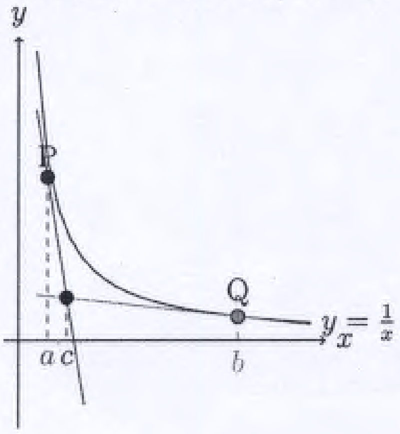
(f) [1] Find the quantity q for which the Marginal Profit is 0.

$$0 = -200q + 300$$

$$200q = 300$$

$$q = \frac{300}{200} = \frac{3}{2}$$

[15] 4. Let $f(x) = \frac{1}{x}$ and c be the x-coordinate of the intersection of the tangent lines going through $P = (a, f(a))$ and $Q = (b, f(b))$ as shown in the diagram.



(a) [5] What is the equation of the tangent line to $y = f(x)$ at the point P?

$\ell_P(x) = f'(a)(x-a) + f(a)$ 2pt $f'(x) = -\frac{1}{x^2}$ 1pt

$\Rightarrow \ell_P(x) = -\frac{1}{a^2}(x-a) + \frac{1}{a}$ 2pt

(b) [5] When $a = 2$ and $b = 3$, calculate c.

$$\left.\begin{array}{ll} (1) & \ell_P(x) = -\frac{1}{2^2}(x-2) + \frac{1}{2} \\ (2) & \ell_Q(x) = -\frac{1}{3^2}(x-3) + \frac{1}{3} \end{array}\right\}$$ 1pt

$-\frac{1}{4}x + \frac{1}{2} + \frac{1}{2} \overset{!}{=} -\frac{1}{9}x + \frac{1}{3} + \frac{1}{3}$ 2pt

$-x + 4 = -\frac{4}{9}x + \frac{8}{3}$

$-9x + 36 = -4x + 24 \Rightarrow -5x = -12$

$c = x = \frac{12}{5}$ 2pt

(c) [5] Show that for any choice of a and b, c is always given by the formula $c = \frac{2ab}{a+b}$ (c is the harmonic mean of a and b).

$-\frac{1}{a^2}(x-a) + \frac{1}{a} \overset{!}{=} -\frac{1}{b^2}(x-b) + \frac{1}{b}$ 2pt

$-b^2(x-a) + ab^2 = -a^2(x-b) + a^2 b$

$-b^2 x + b^2 a + a b^2 = -a^2 x + a^2 b + a^2 b$

$x(a^2 - b^2) = 2ab(a-b)$ 2pt

$x = \frac{2ab(a-b)}{a^2 - b^2} = \frac{2ab(a-b)}{(a+b)(a-b)}$

$\Rightarrow c = x = \frac{2ab}{a+b}$ 1pt