Quiz 2

Math 105 - Section 209

Name: ______________________________

Student Nr.: ______________________________

Question 1 *(5 points)*
Find the critical points of the following function and use the second derivative test to classify them as local minima, local maxima, or saddle points: $f(x, y) = x^4 + 2y^2 - 4xy$.

The first order partial derivatives of $f(x, y)$ are

$$\begin{aligned} \nabla f_x &= 4x^3 - 4y, \\ \nabla f_y &= 4y - 4x. \qquad (1pt) \end{aligned}$$

If both first order partial derivatives vanish, then $x^3 = y$ and $y = x$. Combining the two we get that $x^3 = x$. Then, either $x = 0$ and therefore $y = 0$. Or, if $x \neq 0$, we can divide both sides by x, and we get that $x^2 = 1$. This implies that $x = \pm 1$. All together this means that there are three critical points: $(-1, -1)$, $(0, 0)$ and $(1, 1)$ (*1 pt*). The discriminant of the function is

$$D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - f_{xy}^2(x, y) = (12x^2)(4) - (-4)^2 = 48x^2 - 16. \qquad (1pt)$$

We perform the second derivative test on the three critical points (*2 pts*).

- $D(-1, -1) = 48 - 16 = 32 > 0$, and $f_{xx}(-1, -1) = 12 > 0$, so $(-1, -1)$ is a local minimum.
- $D(0, 0) = -16 < 0$, so $(0, 0)$ is a saddle point.
- $D(1, 1) = 48 - 16 = 32 > 0$, and $f_{xx}(1, 1) = 12 > 0$, so $(1, 1)$ is a local minimum.

Question 2 *(5 points)*
Find the maximum and minimum values of $f(x, y) = y^2 - 6x^2$ subject to the constraint $x^2 + 2y^2 = 2$.

We will use the method of Lagrange Multipliers. The gradients of f and g are:

$$\begin{aligned} \nabla f &= \langle -12x, 2y \rangle \\ \nabla g &= \langle 2x, 4y \rangle \end{aligned}$$

Therefore, the Lagrange Multiplier equations are (*1 pt*):

$$\begin{aligned} &(1) & -12x &= 2\lambda x \\ &(2) & 2y &= 4\lambda y \\ &(3) & x^2 + 2y^2 - 2 &= 0 \end{aligned}$$

Suppose first that $x = 0$. Then, from (3) we get that $2y^2 = 2$, from which it follows that $y = \pm 1$. Then from (2) it follows that $\lambda = \pm\frac{1}{2}$. (*1 pt*)
If $x \neq 0$, then we can divide in (1) with x to get $-12 = 2\lambda$. This gives $\lambda = -6$. Plugging this into (2) gives $2y = -24y$. This can be satisfied only if $y = 0$. Then (3) simplifies to $x^2 = 2$. This gives $x = \pm\sqrt{2}$. (*1 pt*)
The values of f at the possible candidates are (*1 pt*):

$$f(0, \pm 1) = 1$$

$$f\left(\pm\sqrt{2}, 0\right) = -6 \cdot 2 = -12$$

Therefore, the minimum of f subject to the constraint is $--12$, and the maximum of it is 1 (*1 pt*).