Today

- Summary of 2x2 systems all in one picture
- Non-homogeneous systems of ODEs
- Non-homogeneous two-tank example
- Intro to Laplace transforms

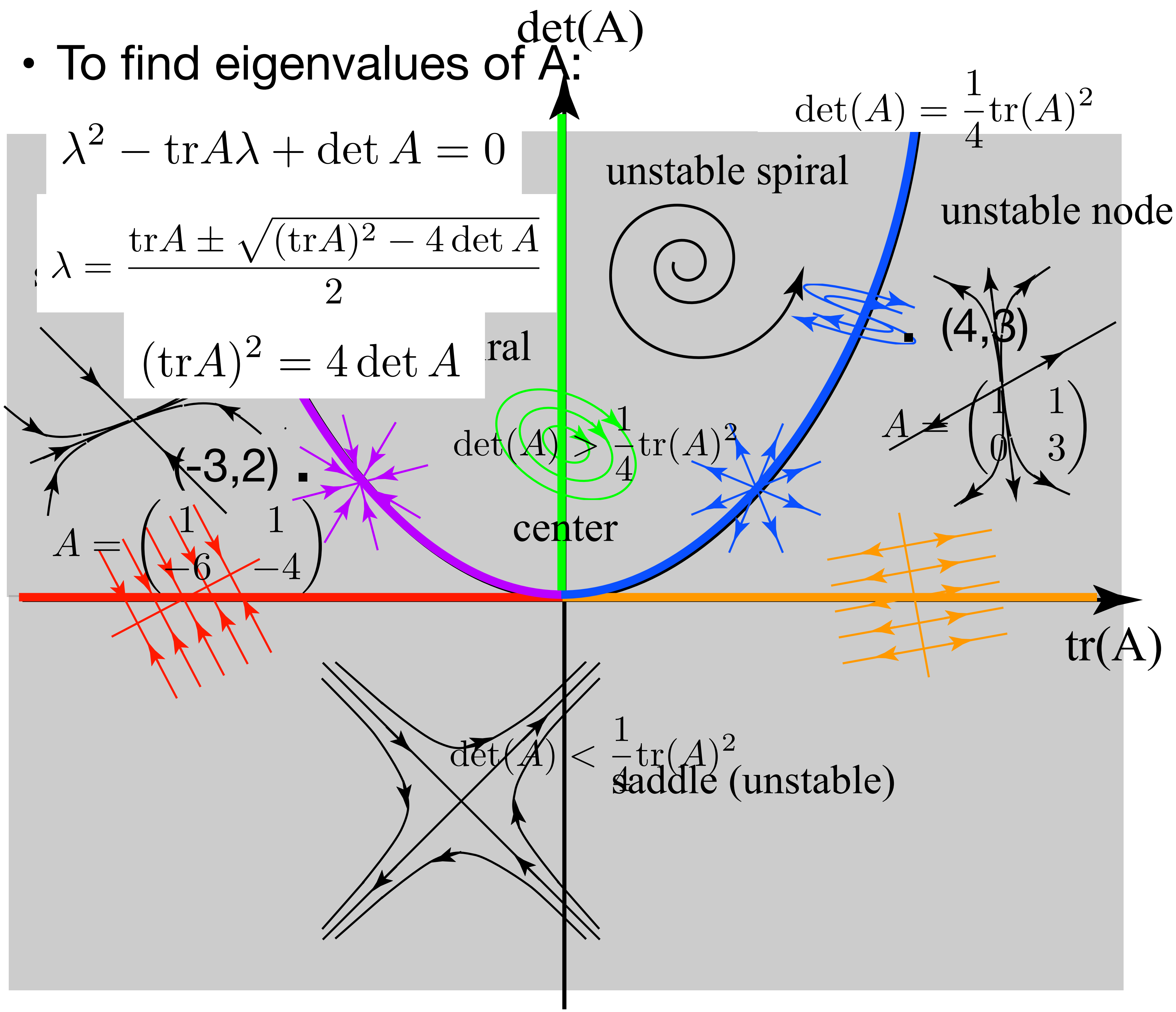
Summary - homogeneous 2x2 systems

- To find eigenvalues of A:

$$\lambda^2 - \mathrm{tr}A\lambda + \det A = 0$$

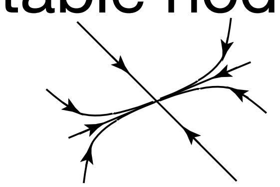
$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}$$

$$(\mathrm{tr}A)^2 = 4\det A$$

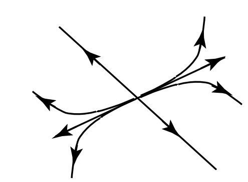


When (tr(A), det(A)) is in the shaded region, we have a…

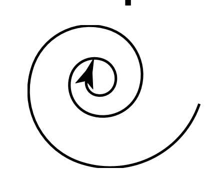
(A) stable node



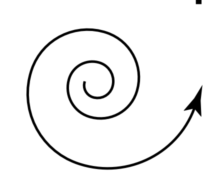
(B) unstable node



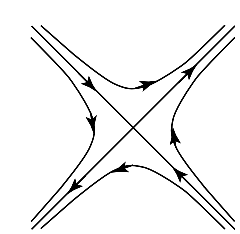
(C) stable spiral



(D) unstable spiral

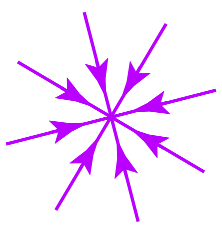


(E) saddle

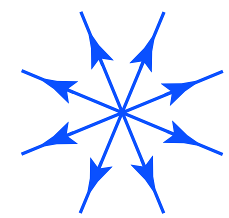


Summary - homogeneous 2x2 systems

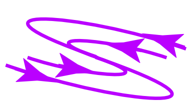
Repeated evalue cases:



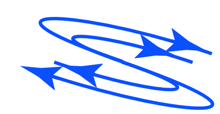
$\lambda<0$, two indep. evectors.



$\lambda>0$, two indep. evectors.

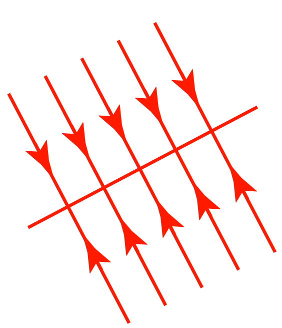


$\lambda<0$, only one evector.

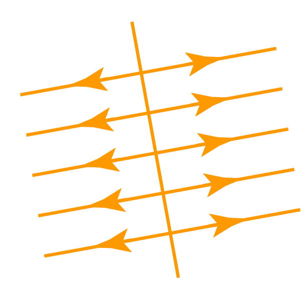


$\lambda>0$, only one evector.

One zero evalue (singular matrix):



$\lambda_1=0$, $\lambda_2<0$,



$\lambda_1=0$, $\lambda_2>0$,

Nonhomogeneous system of DEs

- How do you solve the equation

$$\mathbf{x}'(\mathbf{t}) = A\mathbf{x}(\mathbf{t}) + \mathbf{b} \text{ ?}$$

- Define the linear operator

$$L[\mathbf{x}] = \mathbf{x}'(\mathbf{t}) - A\mathbf{x}(\mathbf{t})$$

- The equation above can be written as

$$L[\mathbf{x}] = \mathbf{b}$$

- As for 2^{nd} order equations, solve homogeneous eqn first,

$$L[\mathbf{x}] = \mathbf{0}$$

- then Method of Undetermined Coefficients...

Nonhomogeneous system of DEs

- For the equation,

$$\mathbf{x}'(\mathbf{t}) = A\mathbf{x}(\mathbf{t}) + \mathbf{b}$$

with …

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{x_p} = \mathbf{v}$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{x_p} = \mathbf{v}$$ (lucky!)

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \mathbf{x_p} = t\mathbf{v} + \mathbf{u}$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \mathbf{x_p} = t\mathbf{v} + \mathbf{u}$$

Nonhomogeneous system of DEs

- For the equation,

$$\mathbf{x}'(\mathbf{t}) = A\mathbf{x}(\mathbf{t}) + \mathbf{b}$$

what form should we guess for $x_p(t)$ (in the sense of MUC)?

(a) $\mathbf{x_p} = \mathbf{v}$ -- works when **b** is in the range of A (which is to say often so try this first, e.g. it always works when A is invertible).

(b) $\mathbf{x_p} = t\mathbf{v}$ -- works when (b) doesn't and **b** happens to be in the nullspace of A which is a special case so safer to go straight from (b) to (d).

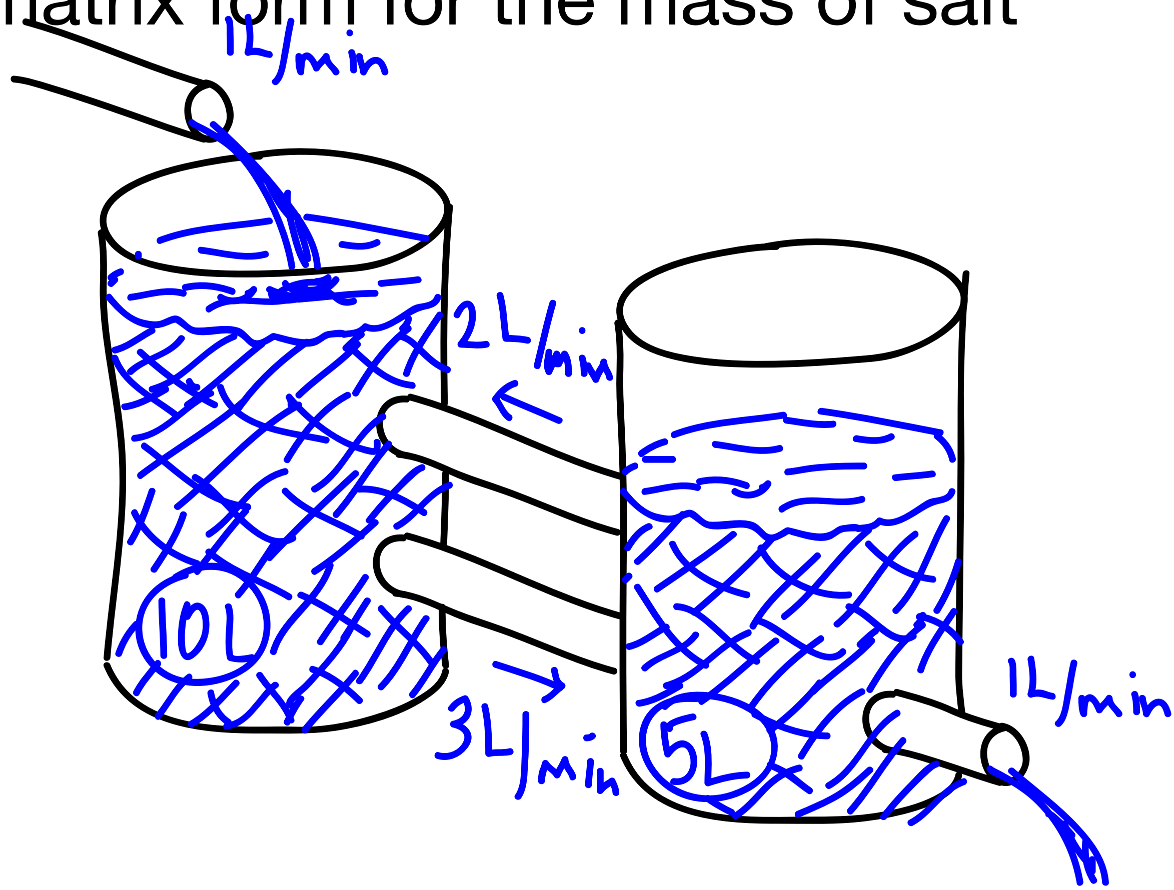
(c) $\mathbf{x_p} = t\mathbf{v} + \mathbf{u}$ -- works when (b) and (c) don't with one exception - when the columns of A and solutions of Av=0 are not independent.

(d) $\mathbf{x_p} = t^2\mathbf{v} + t\mathbf{u} + \mathbf{w}$ -- works when (d) doesn't.

Nonhomogeneous system of DEs - example

- Salt water flows into a tank holding 10 L of water at a rate of 1 L/min with a concentration of 200 g/L. The well-mixed solution flows from that tank into a tank holding 5 L through a pipe at 3 L/min. Another pipe takes the solution in the second tank back into the first at a rate of 2 L/min. Finally, solution drains out of the second tank at a rate of 1 L/min.
- Write down a system of equations in matrix form for the mass of salt in each tank.

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}' = \begin{pmatrix} -\frac{3}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} 200 \\ 0 \end{pmatrix}$$



Nonhomogeneous case - example

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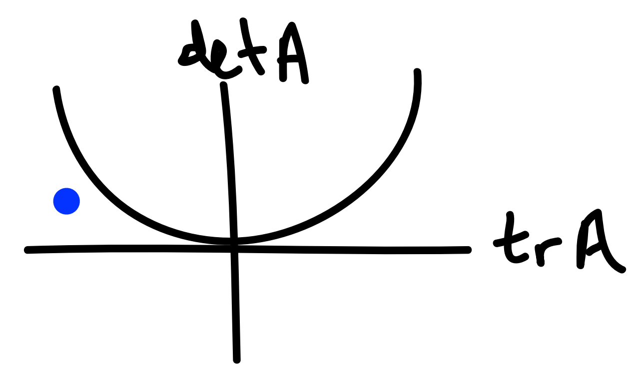
$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}' = \begin{pmatrix} -\frac{3}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} 200 \\ 0 \end{pmatrix}$$

$$\mathrm{tr} A = -\frac{9}{10} \qquad (\mathrm{tr} A)^2 = \frac{81}{100}$$

$$\det A = \frac{9}{50} - \frac{6}{50} = \frac{3}{50} \qquad 4 \det A = \frac{12}{50}$$



Both evalues negative!

Nonhomogeneous case - example

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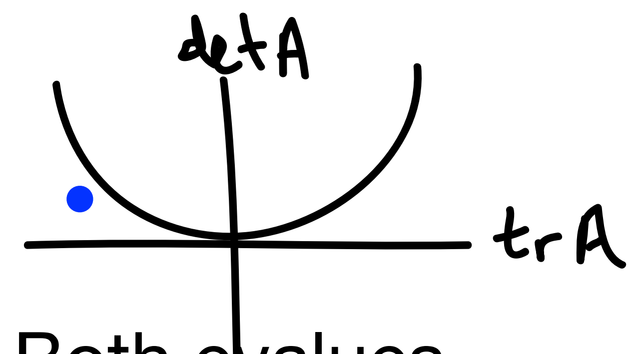

Both evalues negative!

$$\mathbf{m_h}(t) = C_1 e^{\lambda_1 t}\mathbf{v_1} + C_2 e^{\lambda_2 t}\mathbf{v_2} \qquad \left(\lambda_{1,2} = -\frac{9}{20} \pm \frac{\sqrt{57}}{20}\right)$$

$$\mathbf{m_p}(t) = \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\mathbf{0} = A\mathbf{w} + \begin{pmatrix} 200 \\ 0 \end{pmatrix} \longrightarrow A\mathbf{w} = -\begin{pmatrix} 200 \\ 0 \end{pmatrix} \longrightarrow \mathbf{w} = \begin{pmatrix} 2000 \\ 1000 \end{pmatrix}$$

$$\mathbf{m}(t) = C_1 e^{\lambda_1 t}\mathbf{v_1} + C_2 e^{\lambda_2 t}\mathbf{v_2} + \begin{pmatrix} 2000 \\ 1000 \end{pmatrix}$$

Nonhomogeneous case - example

- A “Method of undetermined coefficients” similar to what we saw for second order equations can be used for systems.

- For this course, I’ll only test you on constant nonhomogeneous terms (like the previous example).