Today

- Forced vibrations
  - Newton’s 2nd Law with external forcing.
  - Forced mass-spring system without damping away from resonance.
  - Forced mass-spring system without damping at resonance.
  - Forced mass-spring system with damping.
  - Review questions.

Forced vibrations (3.8)

- Newton's 2nd Law:

$$ma = -kx - \gamma v + F(t)$$

spring force ($-kx$), drag force ($-\gamma v$), applied/external force ($F(t)$)

$$mx'' + \gamma x' + kx = F(t)$$

- Forced vibrations - nonhomogeneous linear equation with constant coefficients.

- Building during earthquake, tuning fork near instrument, car over washboard road, polar bond under EM stimulus (classical, not quantum).

Forced vibrations (3.8)

- Without damping ($\gamma = 0$).

forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

$$x_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

natural frequency

- Case 1: $\omega \neq \omega_0$

$$x_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$x_p''(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$$

$$mx_p'' + kx_p = (k - \omega^2 m)A\cos(\omega t) + (k - \omega^2 m)B\sin(\omega t)$$

$$= F_0\cos(\omega t) \;\Rightarrow A = \frac{F_0}{(k - \omega^2 m)}, B = 0$$

Forced vibrations (3.8)

- Case 2: $\omega = \omega_0$

$$mx'' + kx = F_0 \cos(\omega t)$$

$$x'' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$x_p(t) = t(A\cos(\omega_0 t) + B\sin(\omega_0 t))$$

$$x_p'(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$
$$+t(-\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t))$$

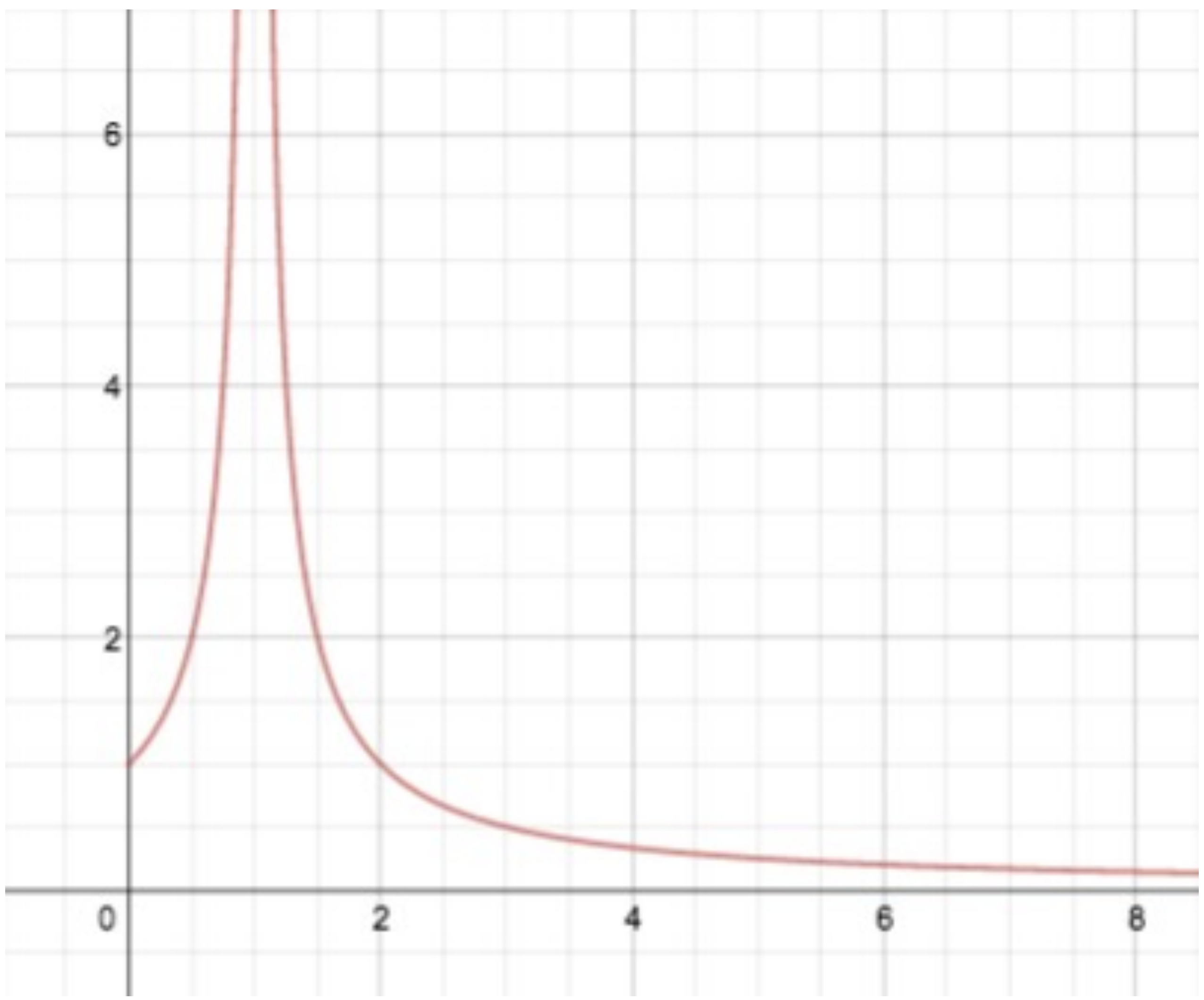
$$x_p''(t) = -\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t)$$
$$+(-\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t))$$
$$+t(-\omega_0^2 A\cos(\omega_0 t) - \omega_0^2 B\sin(\omega_0 t))$$

$$A = 0$$

$$B = \frac{F_0}{2\omega_0 m} = \frac{F_0}{2\sqrt{km}} \qquad x_p(t) = \frac{F_0}{2\sqrt{km}} t\sin(\omega_0 t)$$

Forced vibrations (3.8)

- Plot of the amplitude of the particular solution as a function of ω.



- Calculated:

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

- Plotted:

$$A(\omega) = \frac{1}{|\omega_0^2 - \omega^2|}$$

- Recall that for $\omega = \omega_0$, the amplitude grows without bound.

Forced vibrations (3.8)

- With damping (on the blackboard)

Review questions

Prob. 2. (3 pts.) Here are three *nonlinear* differential equations. Circle *all* the terms that make them *nonlinear*.

(i) $y'' + t\,yy' + y^2 - t^2 = 0$

(ii) $y' + t\sin(y) = 5\,ty$

(iii) $y' + y\sin(t) = 5(ty)^2$

Review questions

- A dye diffuses between two chambers at a rate proportional to the difference in concentrations (c_1 and c_2) between the chambers (with proportionality constant $k>0$). Write down a differential equation for the concentration in the first chamber.

$$\frac{dc_1}{dt} = k(c_2 - c_1)$$

- Solve: $y' - 2ty = t$

$$y(t) = Ce^{t^2} - \frac{1}{2}$$

Review questions

Prob. 2. (6 pts.)

Consider the equation for a linear oscillator with frequency = 2:

$$d^2y/dt^2 + 4y = 0: \quad y(0) = 2, \; y'(0) = 4.$$

Express the solution in the form y = R cos(2t - ϕ), i.e. solve this initial value problem and find R and ϕ.

$$y(t) = 2\sin(2t) + 2\cos(2t)$$

$$A = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$y(t) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(2t) + \frac{1}{\sqrt{2}}\cos(2t)\right)$$

$$\sin\phi = \frac{1}{\sqrt{2}}, \quad \cos\phi = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \phi = \frac{\pi}{4}$$

$$y(t) = 2\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)$$

Review questions

Prob. 2. (6 pts.)
Consider the equation for a linear oscillator with frequency = 2:

$$d^2y/dt^2 + 4y = 0: \quad y(0) = 2, \; y'(0) = 4.$$

Express the solution in the form y = R cos(2t - ϕ), i.e. solve this initial value problem and find R and ϕ.

Note that to convert from $y(t) = 2\sin(2t) + 2\cos(2t)$

to a single cos function we used the identity

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

with $A = 2t$ and $B = -\phi$ so

$$\cos(2t - \phi) = \cos(2t)\cos(-\phi) - \sin(2t)\sin(-\phi)$$

or equivalently

$$\cos(2t - \phi) = \cos(2t)\cos(\phi) + \sin(2t)\sin(\phi)$$