Today

- Fourier series calculations

Fourier series

- Calculate the coefficients of the Fourier series of a function:

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$
$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

$$v_0(x) = 1$$
$$v_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$
$$w_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$f_{FS}(x) = \frac{a_0}{2} v_0(x) + a_1 v_1(x) + a_2 v_2(x) + \cdots$$
$$+ b_1 w_1(x) + b_2 w_2(x) + \cdots$$

$$f_{FS}(x) \circ v_n(x) = \frac{a_0}{2} \cancelto{0}{v_0(x) \circ v_n(x)} + a_1 \cancelto{0}{v_1(x) \circ v_n(x)} + a_2 \cancelto{0}{v_2(x) \circ v_n(x)} + \cdots$$
$$+ b_1 \cancelto{0}{w_1(x) \circ v_n(x)} + b_2 \cancelto{0}{w_2(x) \circ v_n(x)} + \cdots$$
$$= a_n v_n(x) \circ v_n(x) = a_n L$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f_{FS}(x) \cos\left(\frac{n\pi x}{L}\right)\ dx$$

Fourier series

- Calculate the coefficients.

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$
$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

$$a_0 = \int_{-1}^{1} f(x)\ dx$$

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\ dx$$

$$b_n = \int_{-1}^{1} f(x) \sin(n\pi x)\ dx$$

$a_0 =$ / $b_n =$

★ (A) 0

(B) $\frac{1}{2}$ / (B) $\frac{2}{n\pi}$

(C) undefined

(C) $\frac{4(-1)^n}{n\pi}$

(D) $\frac{1}{n\pi}$

★ (D) $\frac{2(1-(-1)^n)}{n\pi}$

$$b_n = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

$$f_{FS}(x) = \frac{4}{\pi}\sin\left(\frac{\pi x}{L}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi x}{L}\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi x}{L}\right) + \cdots$$

https://www.desmos.com/calculator/tlvtikmi0y

Does f(x) = f_{FS}(x) for all x?

Problems at jumps! x=-1, 0, 1

Fourier series

- **Theorem** Suppose f anf f' are piecewise continuous on [-L,L] and periodic beyond that interval. Then $f(x) = f_{FS}(x)$ at all points at which f is continuous. Furthermore, at points of discontinuity, $f_{FS}(x)$ takes the value of the midpoint of the jump. That is,

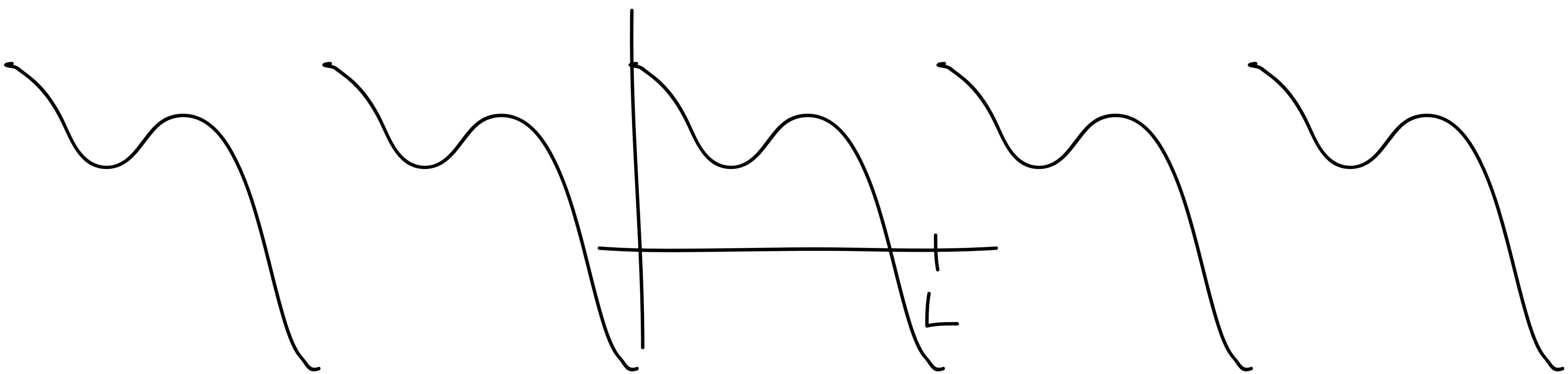
$$f_{FS}(x) = \frac{f(x^+) + f(x^-)}{2}$$

Fourier series

- Suppose you have a function on the interval [0,L] and you would like to represent it using a Fourier series. Need to make it periodic somehow. There are a few options for how to do this.
    1. Use the function given on [0,L] and extend it outside that interval so that it has period L.

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$
$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

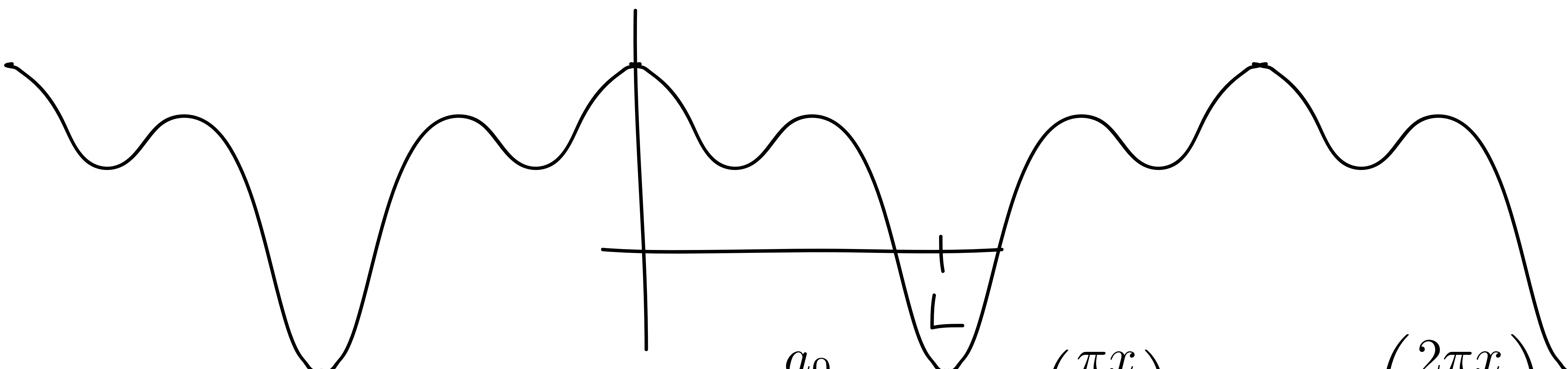
- Is this extension even? odd? Neither!



Fourier series

- Suppose you have a function on the interval [0,L] and you would like to represent it using a Fourier series. Need to make it periodic somehow. There are a few options for how to do this.
  1. Use the function given on [0,L] and extend it outside that interval so that it has period L.
  2. Reflect it about y-axis first, then extend with period 2L.

    - Is this extension even? odd? Even!

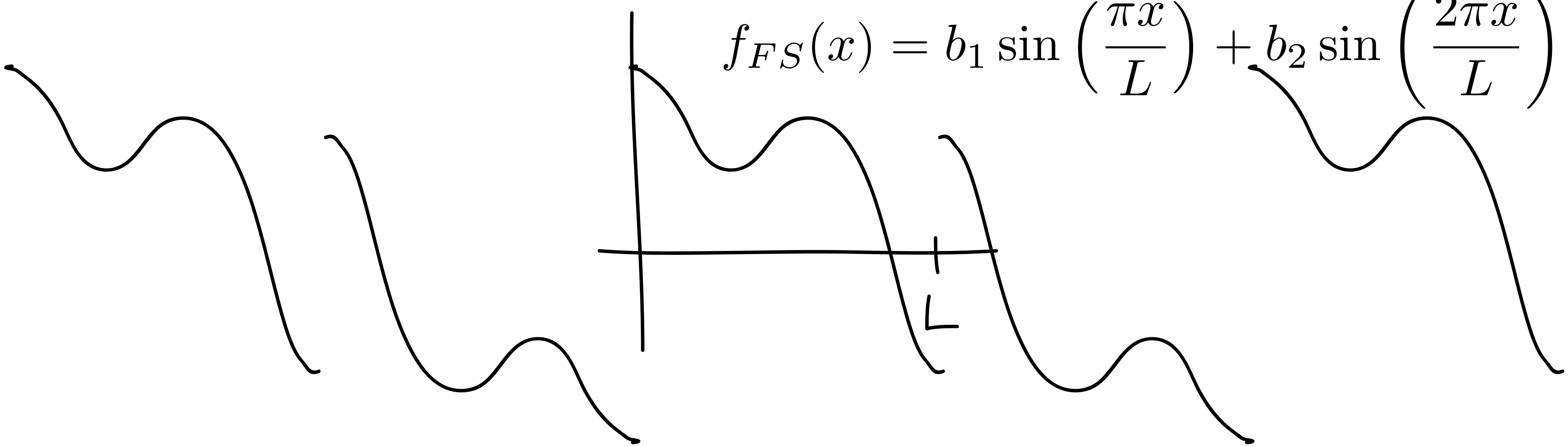


$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$

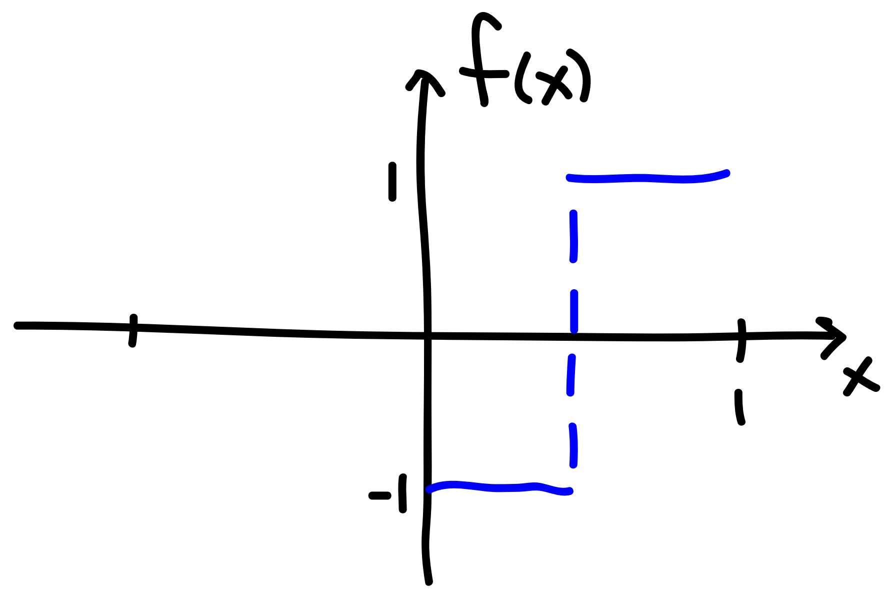
Fourier series

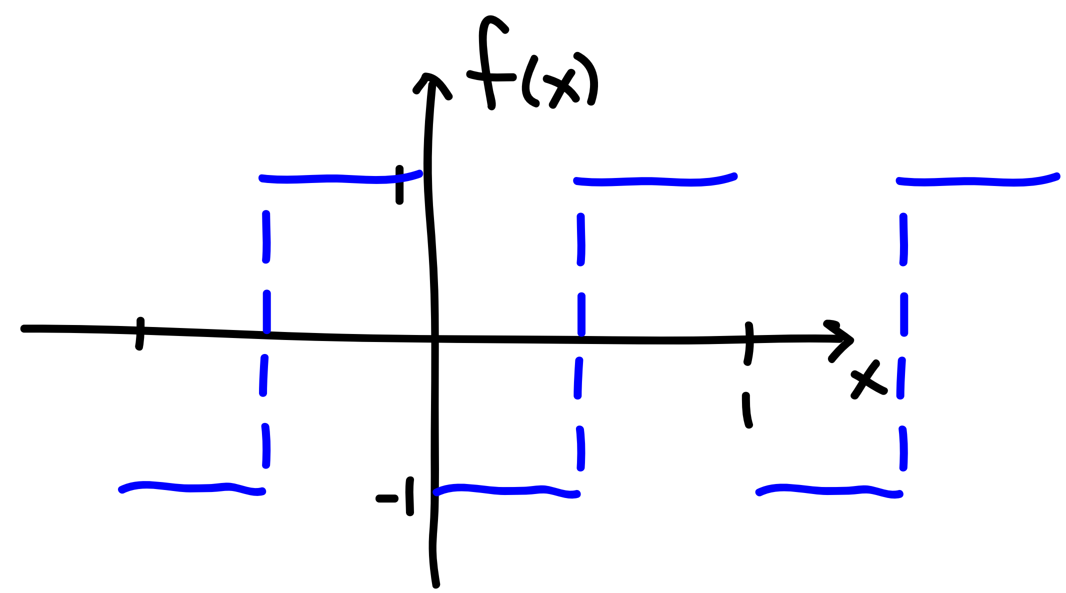
- Suppose you have a function on the interval [0,L] and you would like to represent it using a Fourier series. Need to make it periodic somehow. There are a few options for how to do this.
    1. Use the function given on [0,L] and extend it outside that interval so that it has period L.
    2. Reflect it about y-axis first, then extend with period 2L.
    3. Rotate it about origin, then extend with period 2L.
        - Is this extension even? odd? Odd!

$$f_{FS}(x) = b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

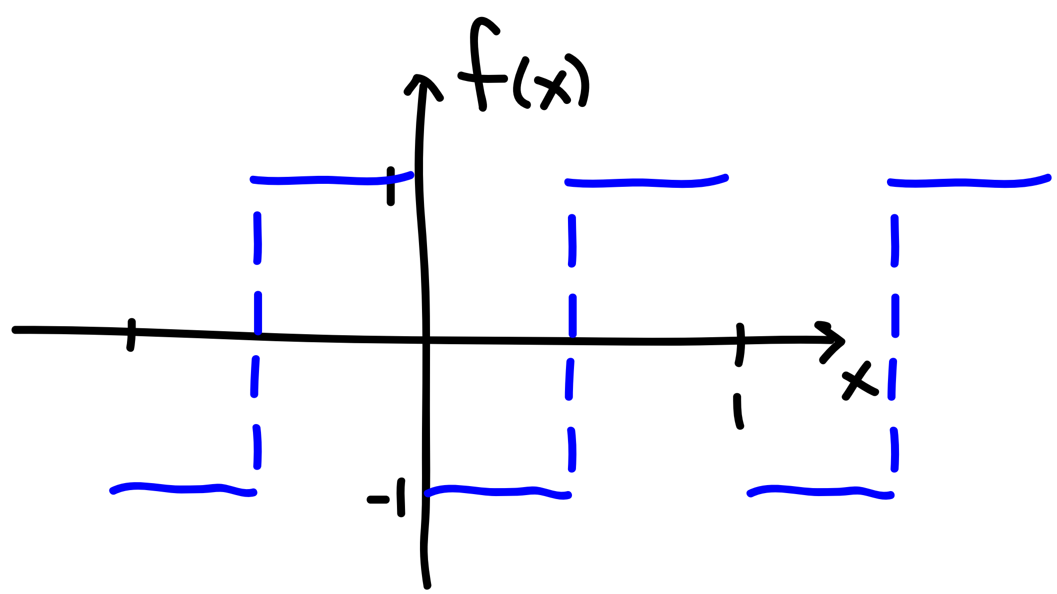


Examples

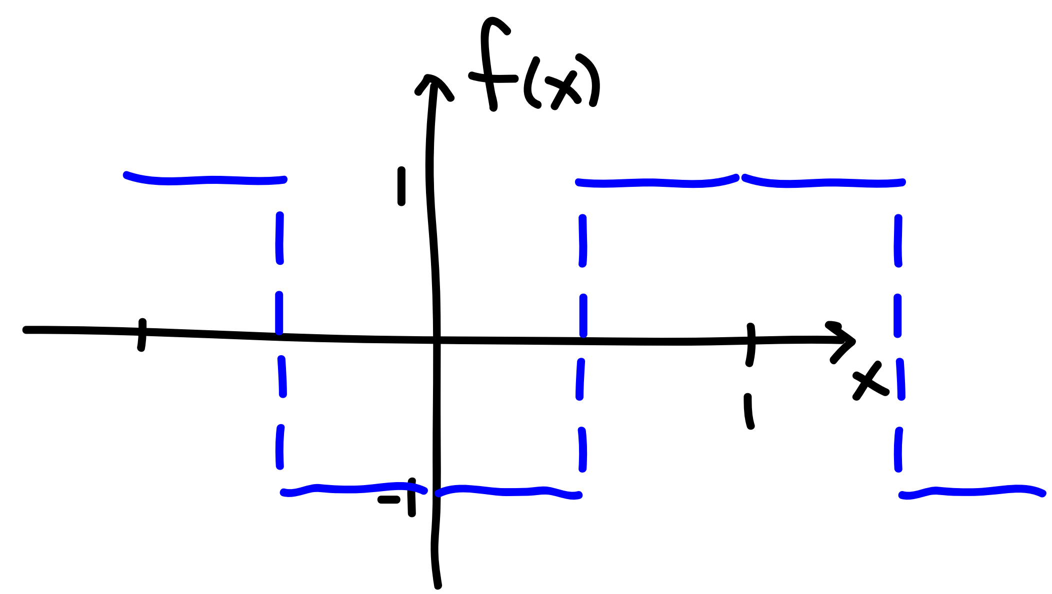




Odd periodic extension



Periodic extension



Even periodic extension