Today

- Reminders:
  - Pre-lecture assignment for Thursday 7 am
  - Week 1 assignment due Friday 5 pm.
- Finish up separating variables
- Modeling
- Existence and uniqueness (not going to test on the theory)

Separable equations (Section 2.2)

- What is $\frac{d}{dt}e^y$?

(A) e^y

(B) $e^y\frac{dy}{dt}$

(C) ye^{y-1}

(D) $ye^{y-1}\frac{dy}{dt}$

(E) Don't know.

- Solve $\frac{dy}{dt} = e^{-y}t^2$.

(A) $y(t) = t^2e^t + C$

(B) $y(t) = \frac{1}{3}t^3 + C$

(C) $y(t) = \ln\left(\frac{1}{3}t^3\right) + C$

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- Recognize a chain rule: $\dfrac{d}{dx}(G(y)) = G'(y)\dfrac{dy}{dx}$.
- Take antiderivatives to get $G(y) = F(x) + C$.
- Finally, solve for y if possible: $y(x) = G^{-1}(F(x) + C)$.

Separable equations (Section 2.2)

- Solve: $$\frac{dy}{dx} = -\frac{x}{y}$$

(A) $y(x) = x$

(B) $y(x) = \sqrt{C - x^2}$

(C) $y(x) = \sqrt{x^2 + C}$

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(E) None of these (or don't know)

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Does (B) cover all possible initial conditions?

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- Or express implicitly: $y^2 = -x^2 + C$
- To satisfy an IC, must choose a value for C and + or - .

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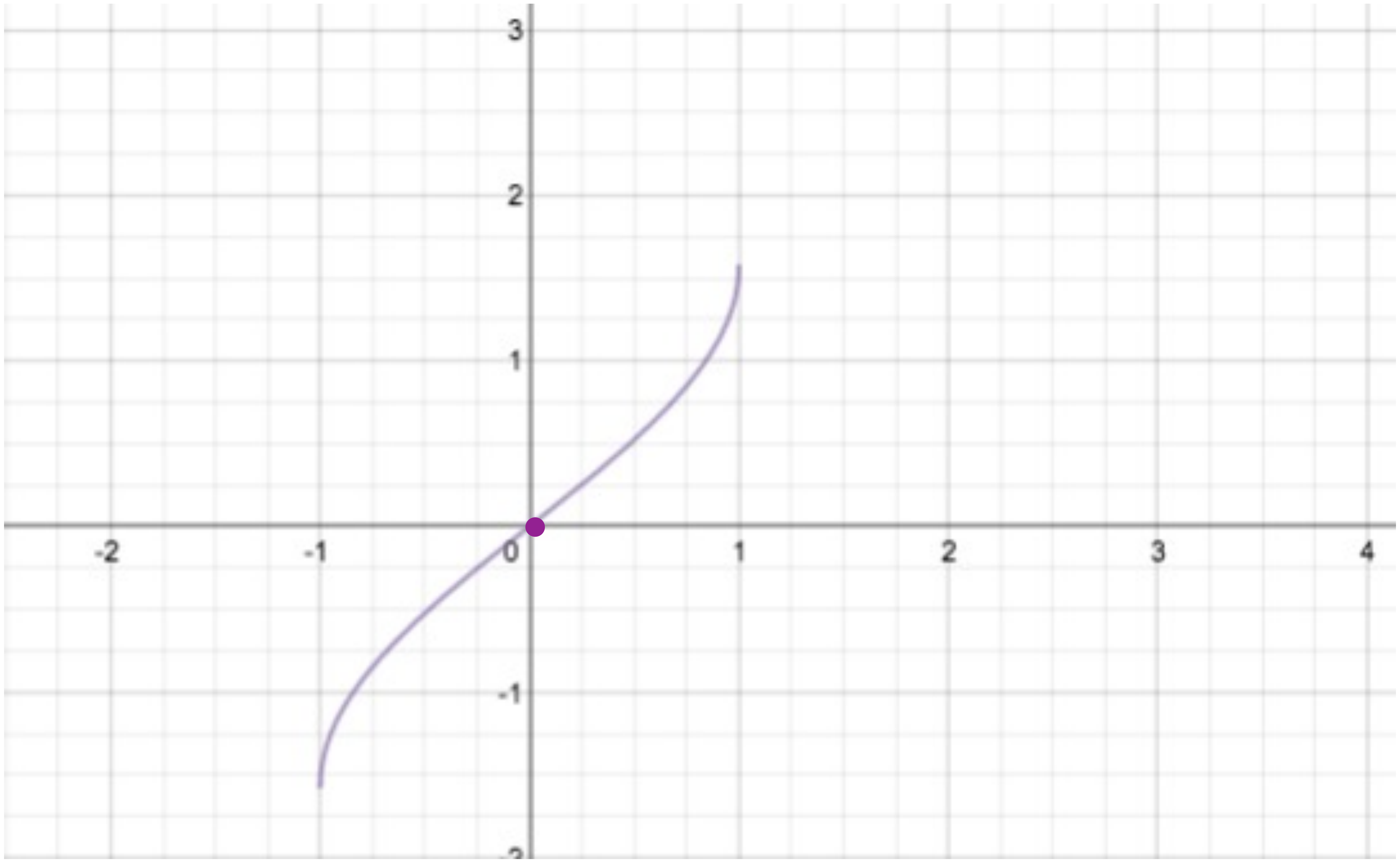

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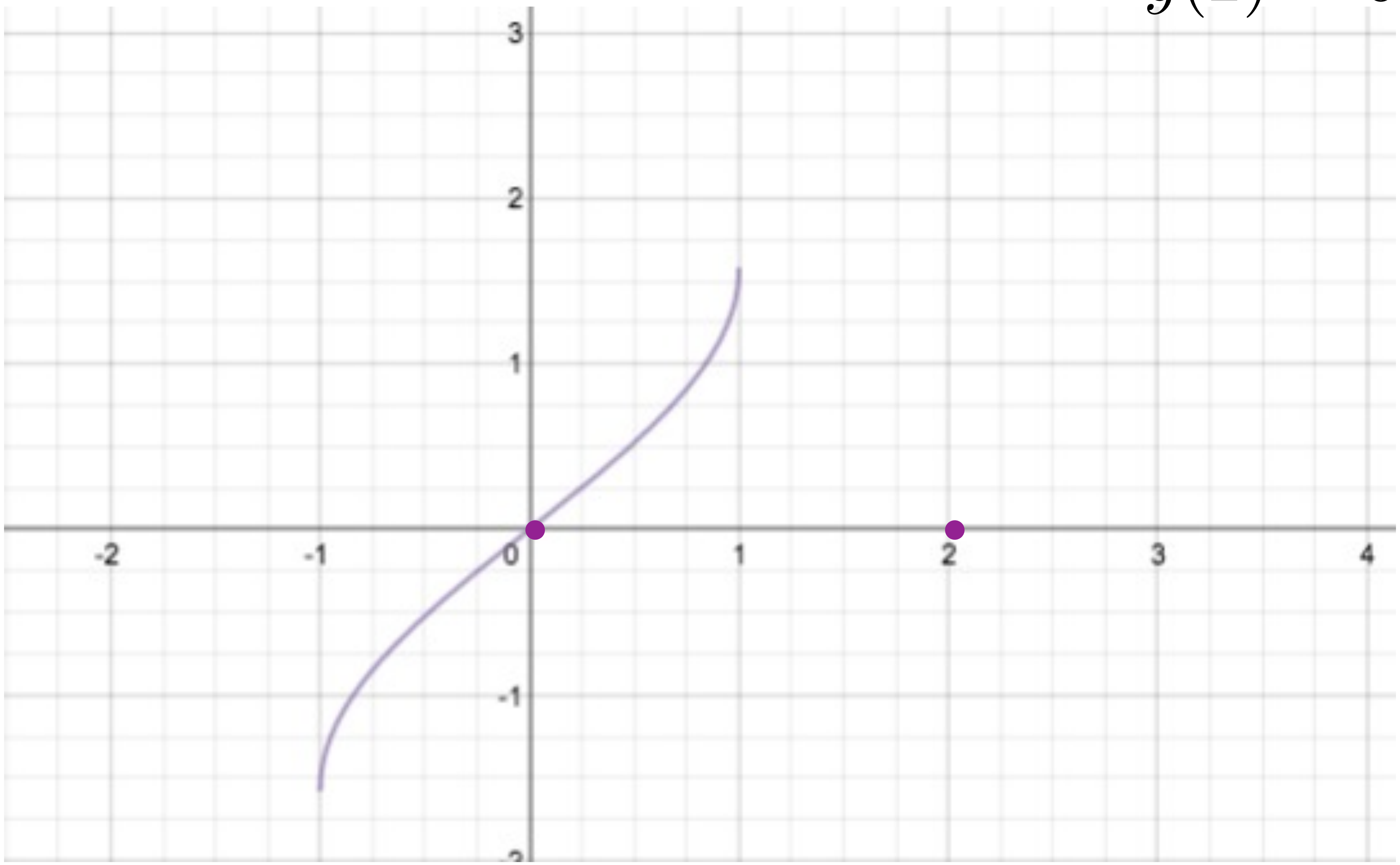

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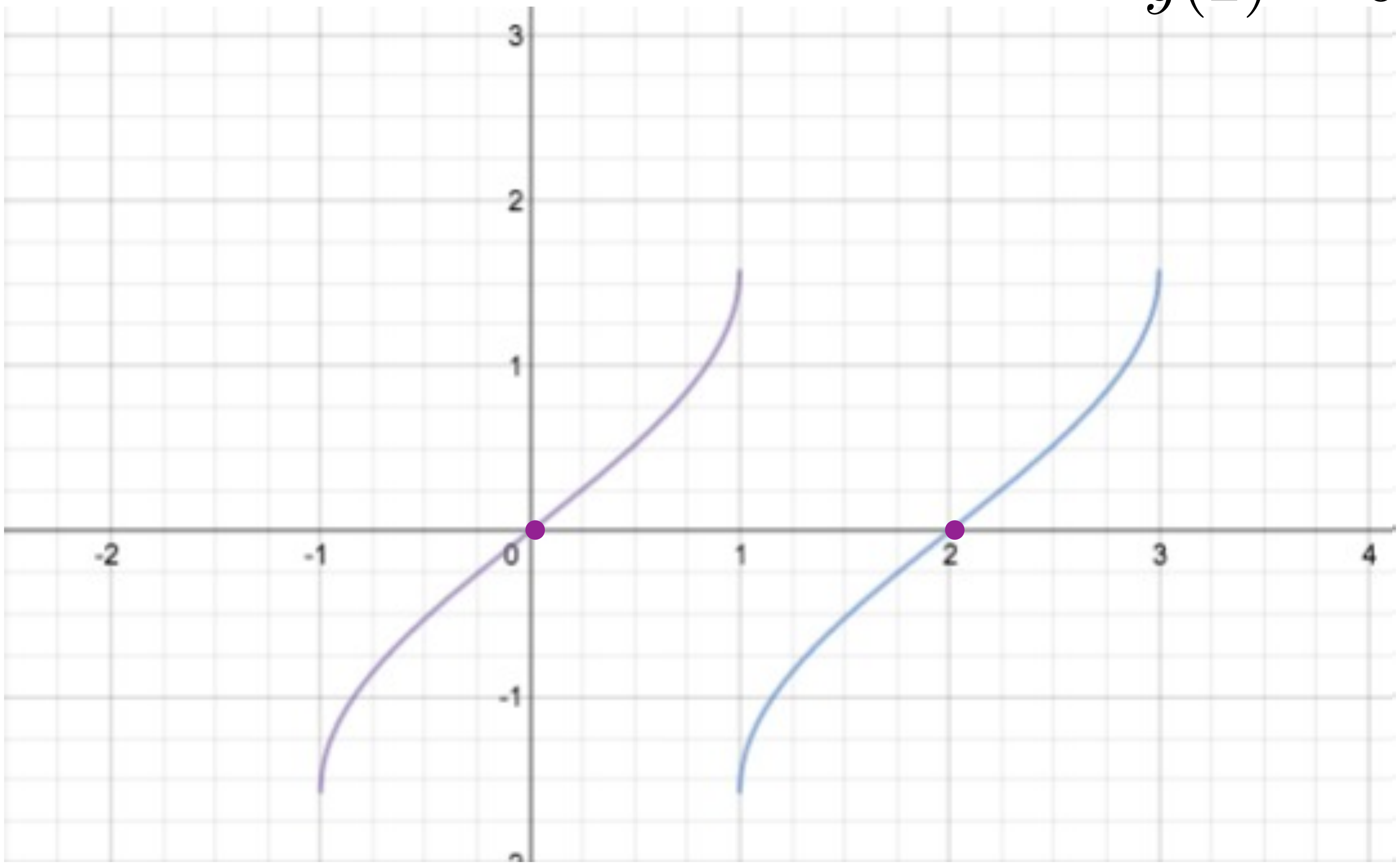

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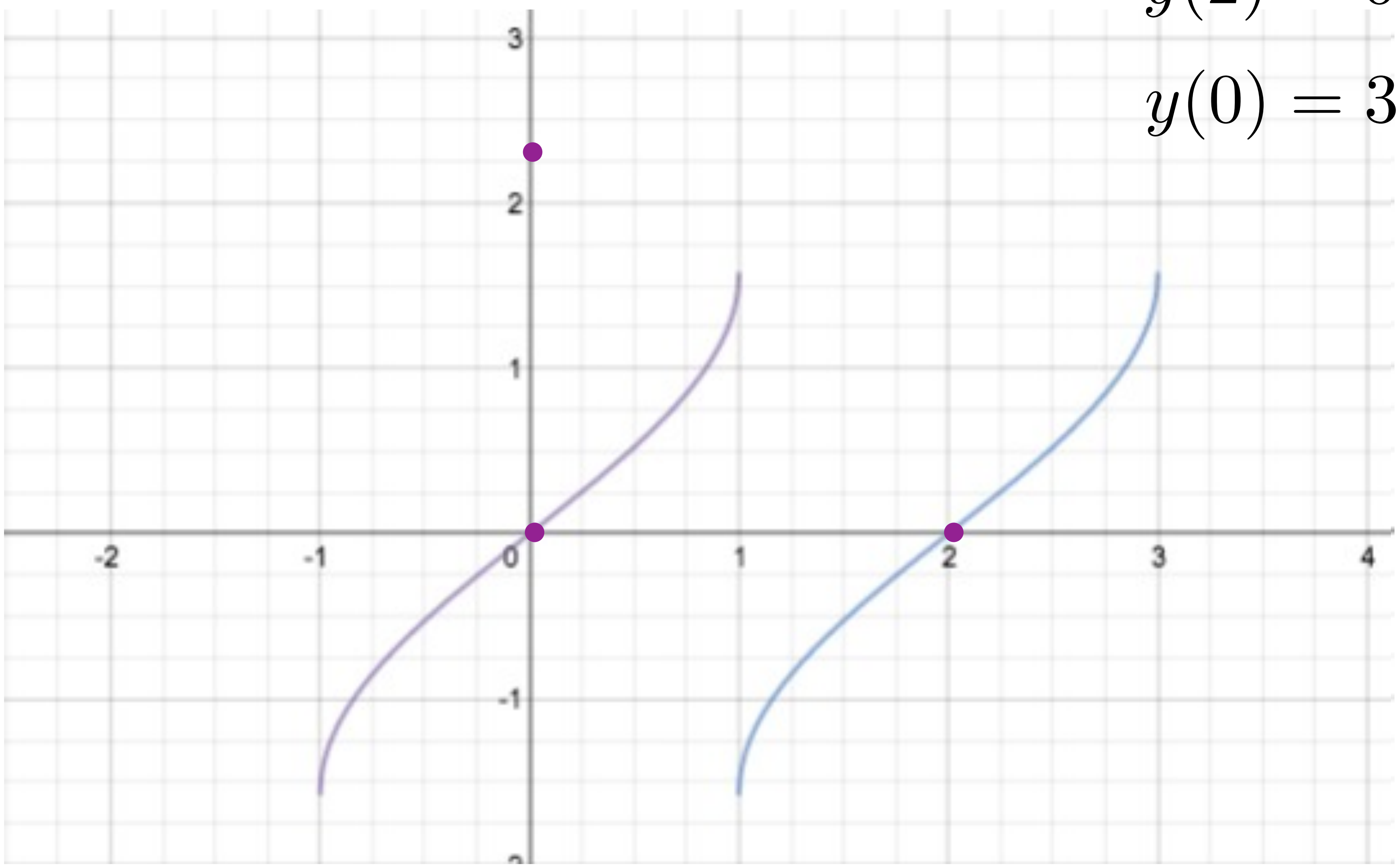

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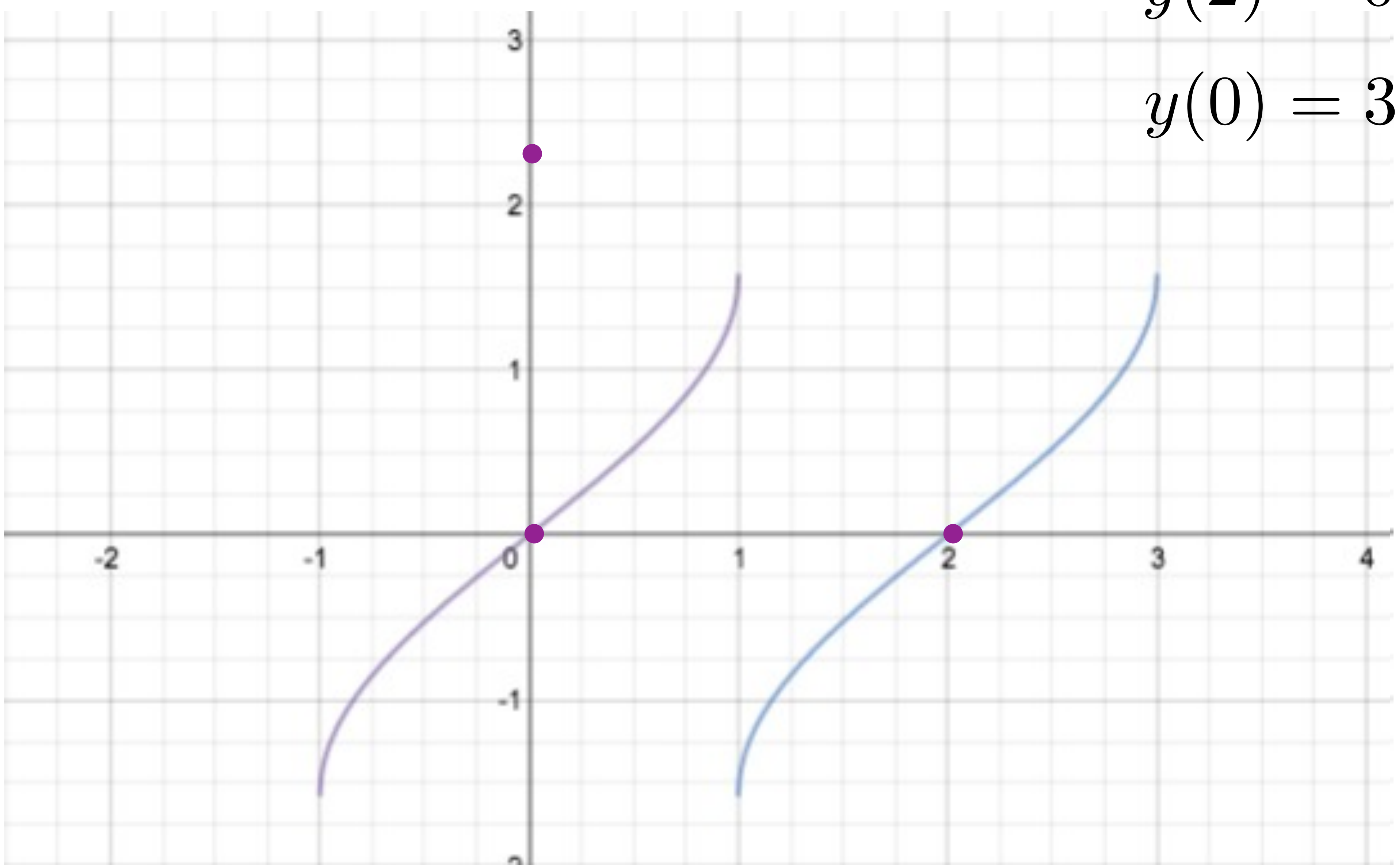

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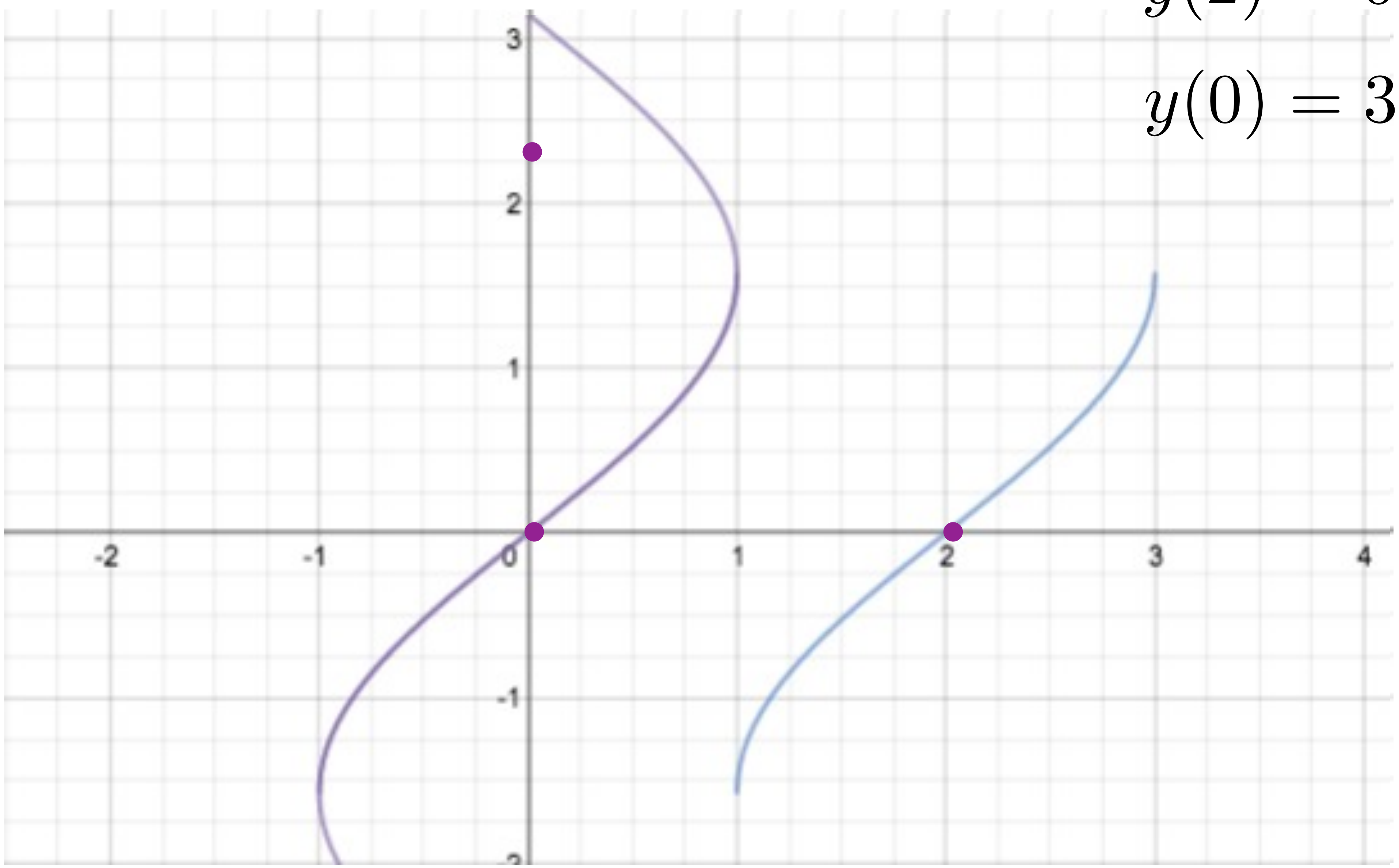

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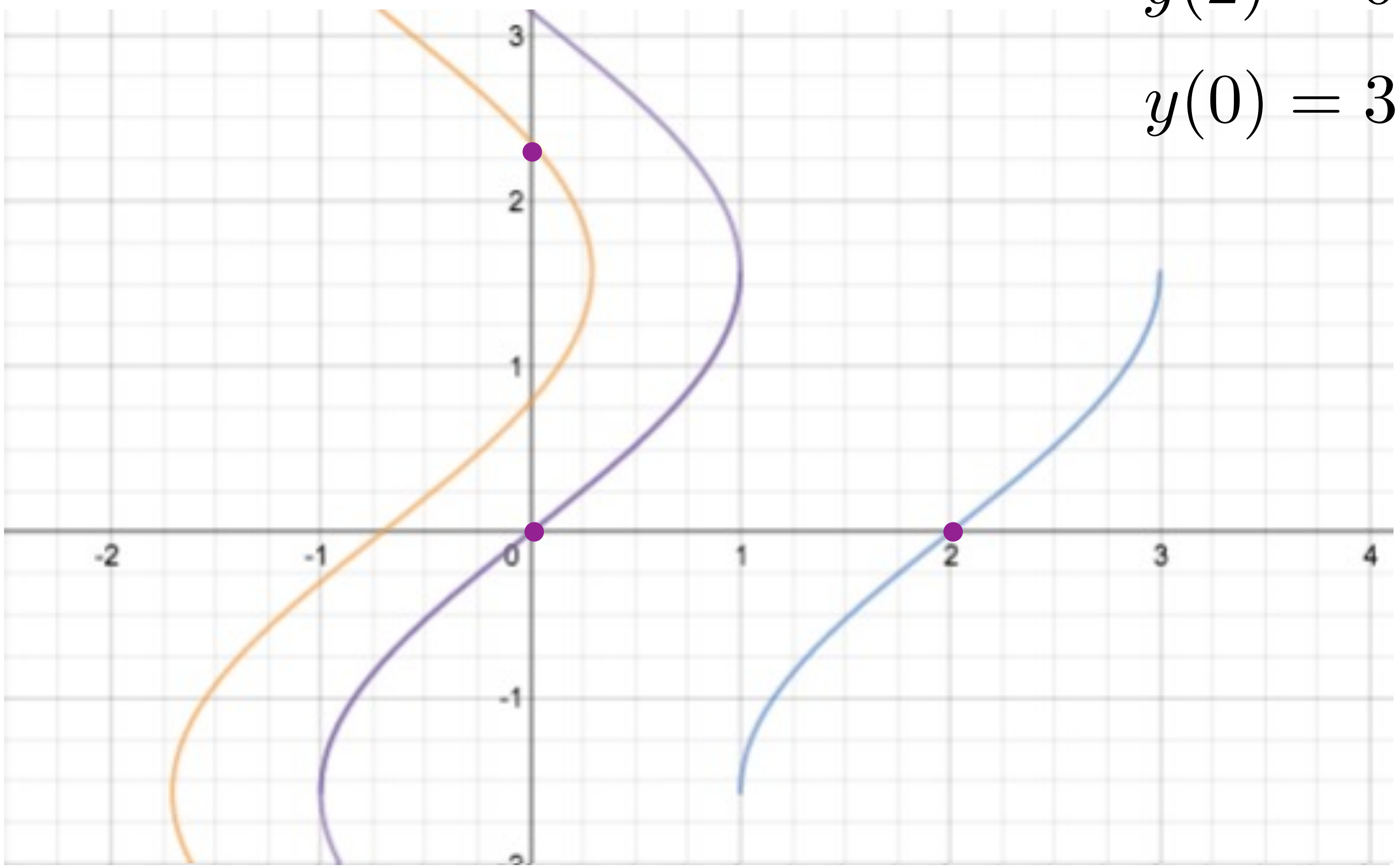

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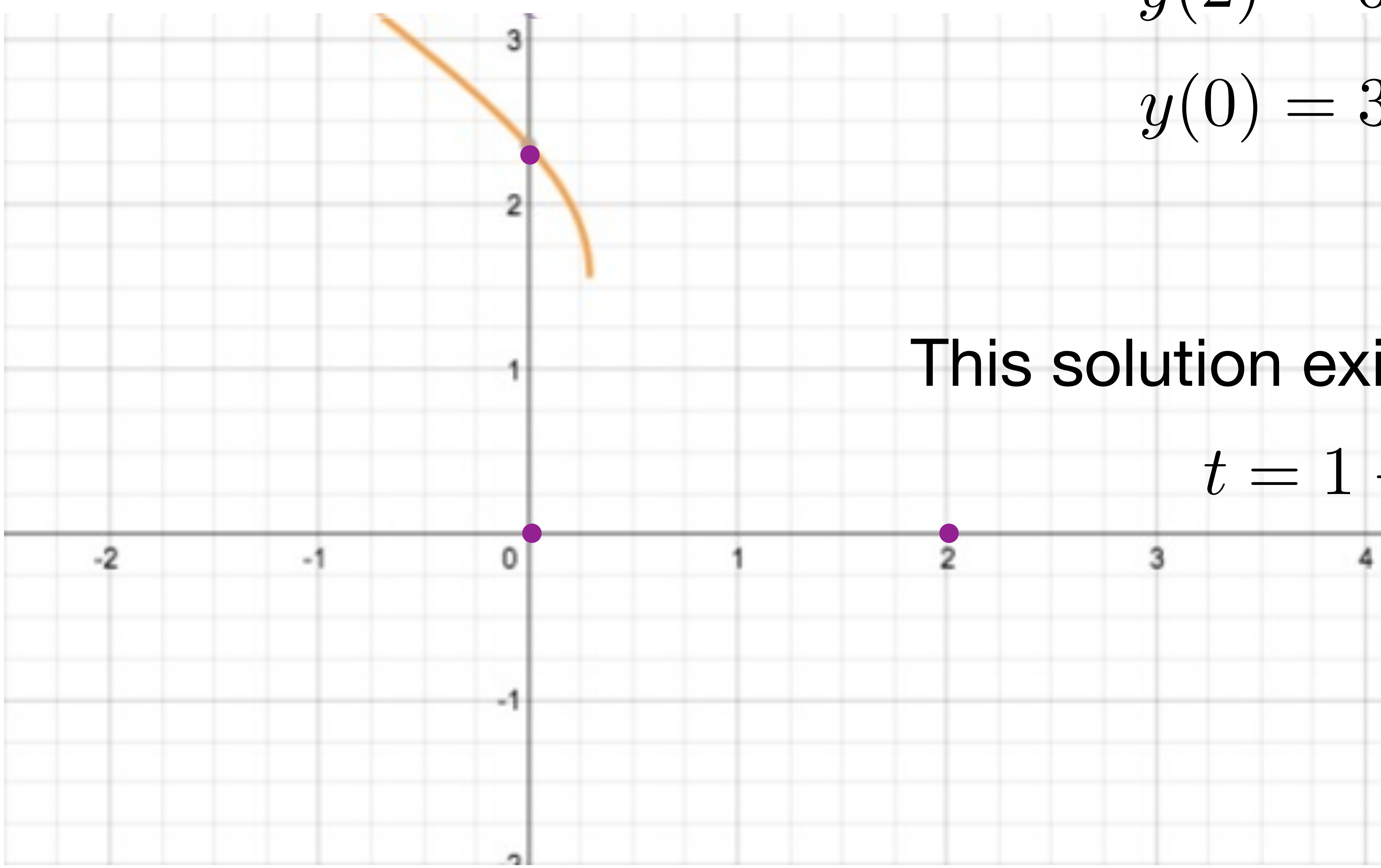

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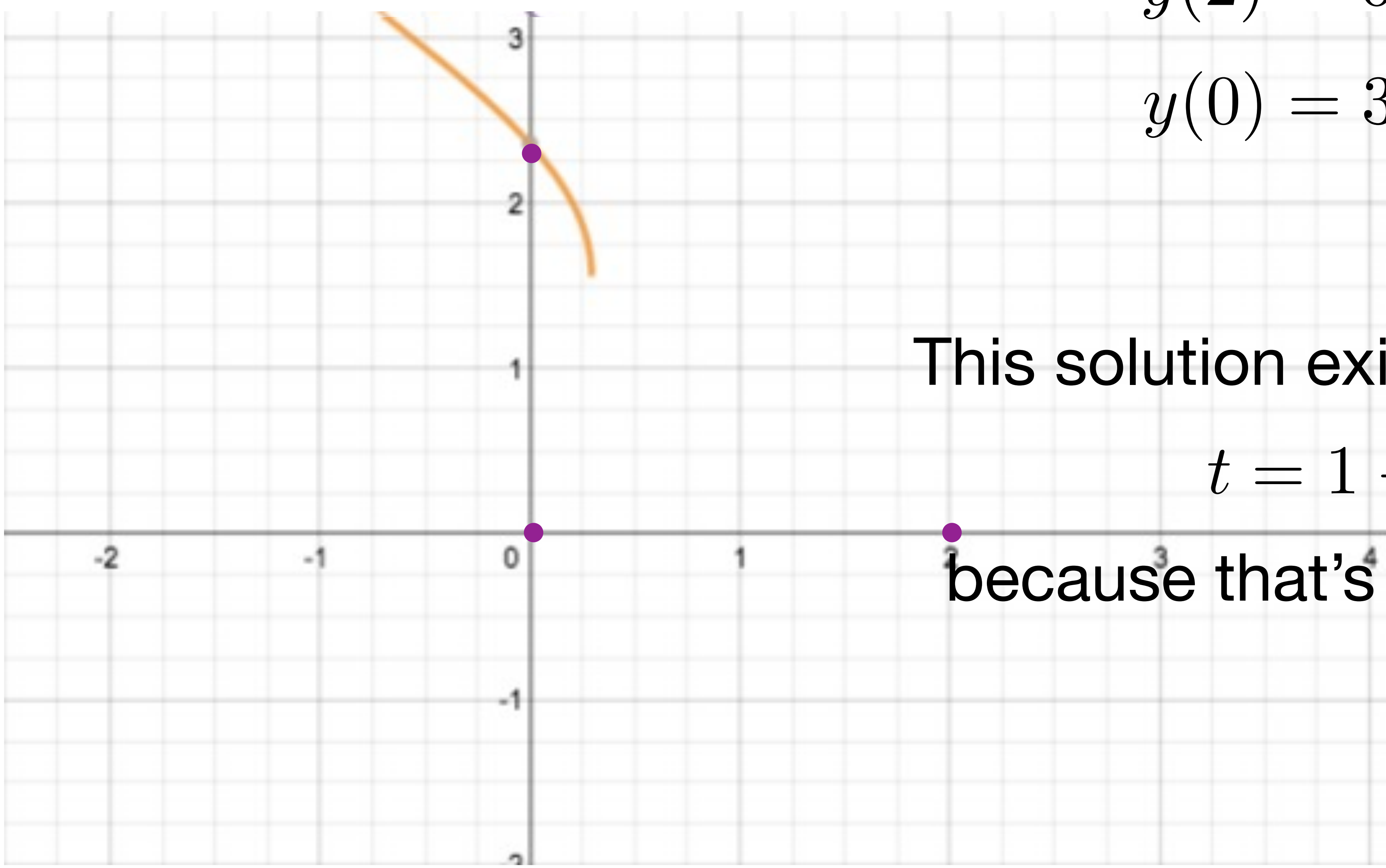

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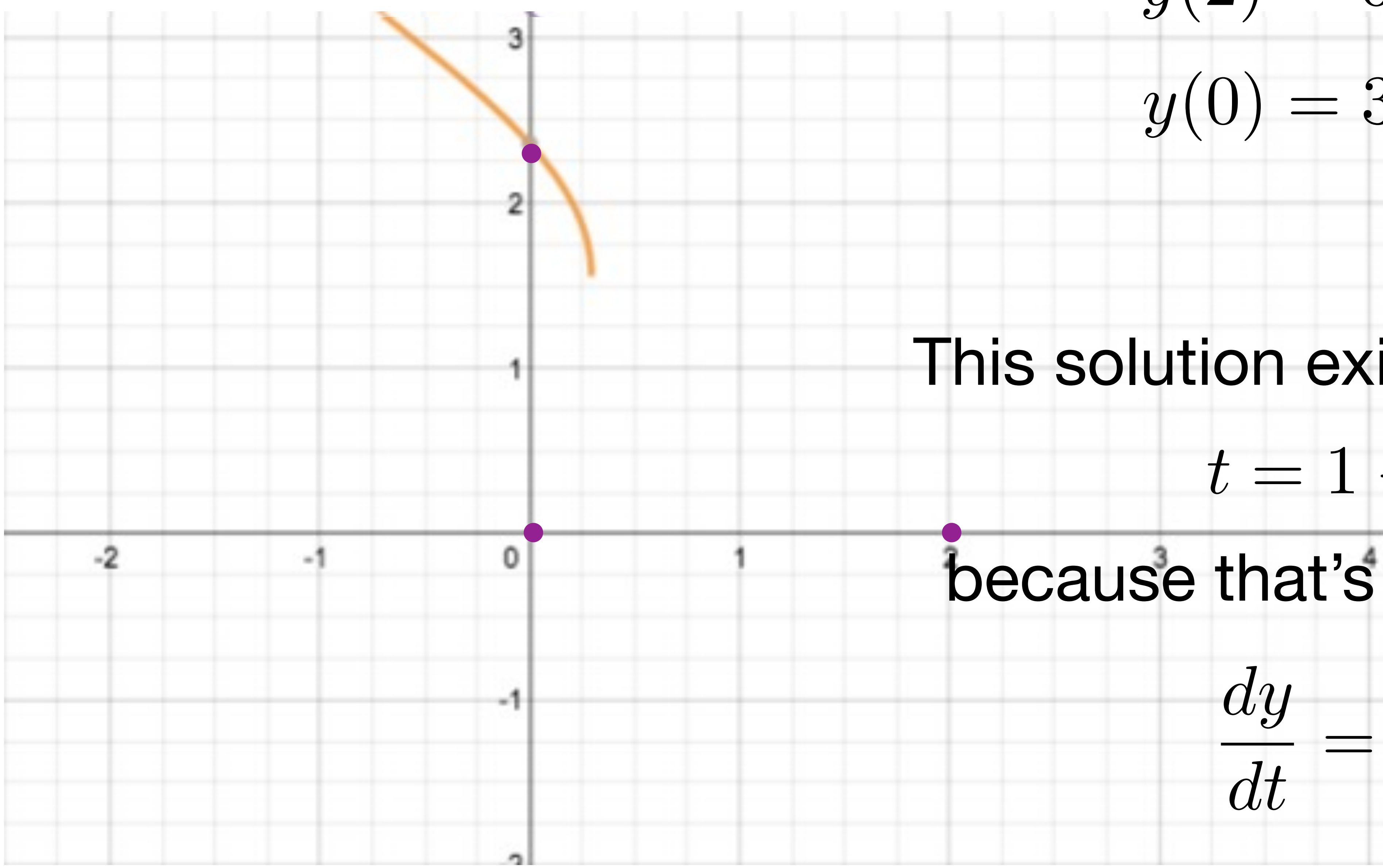

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  - Take limit as $\Delta t \to 0$ to get an equation for q(t).

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- Freshwater flows into a tank at a rate 2 L/min. The tank starts with a concentration of 100 g / L of salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

    (a) Write down an IVP for the mass of salt in the tank as a function of time.

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(A) $\Delta m \approx -2 \text{ L/min} \times m(t) \ / \ 10 \text{ L}$

(B) $\Delta m \approx -2 \text{ L/min} \times 100 \text{ g/L} \times \Delta t$

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- Rearranging: $\dfrac{m(t+\Delta t) - m(t)}{\Delta t} \approx -\dfrac{1}{5}m(t)$

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- $m(t + \Delta t) \approx m(t) - \Delta t \ \times \ 2 \text{ L/min} \ \times \ m(t) \ / \ 10 \text{ L}$
- Rearranging: $\dfrac{m(t + \Delta t) - m(t)}{\Delta t} \approx -\dfrac{1}{5} m(t)$
- Finally, taking a limit:

$$\boxed{\frac{dm}{dt} = -\frac{1}{5} m(t)}$$

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- m(0) = 1000 g.

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To think about: what is the most general equation that can be solved using (A) and (B)?

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$$\lim_{t\to\infty} m(t) = 0$$

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- Saltwater with a concentration of 200 g/L flows into a tank at a rate 2 L/min. The tank starts with no salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

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- What happens when $m > 2000$? ---> $m' < 0$.
- Limiting mass: 2000 g (Long way: solve the eq. and let $t \to \infty$.)

Existence and uniqueness (Section 2.4)

Theorem 2.4.2 Let the functions f and $\dfrac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta, \quad \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t_0 < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the IVP

$$y' = f(t, y), \quad y(t_0) = y_0.$$

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- A couple questions/examples to explore on your own:
  - Why don't we get a solution all the way to the ends of the t interval?
    - Example: $$\frac{dy}{dt} = y^2, \quad y(0) = 1$$
  - How does a non-continuous RHS lead to more than one solution?
    - Example: $$\frac{dy}{dt} = \sqrt{y}, \quad y(0) = 0$$

Second order linear equations (Chapter 3)

- The general form for a second order linear equation:

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- As with first order linear equations, we have homogeneous (g=0) and non-homogeneous second order linear equations.

- We'll start by considering the homogeneous case with constant coefficients:

$$ay'' + by' + cy = 0$$

Homog. eq. with constant coeff. (Section 3.1)

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- Instead, find two independent solutions, $y_1(t)$, $y_2(t)$, by whatever method.
- The general solution will be $y(t) = C_1y_1(t) + C_2y_2(t)$.