Today

- Non-homogeneous systems of ODEs
- Non-homogeneous two-tank example
- Intro to Laplace transforms

Nonhomogeneous system of DEs

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- then Method of Undetermined Coefficients...

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which of the following is a suitable guess (in the sense of MUC)?

(A) $\mathbf{x_p} = c\mathbf{b}$

(B) $\mathbf{x_p} = \mathbf{v}$

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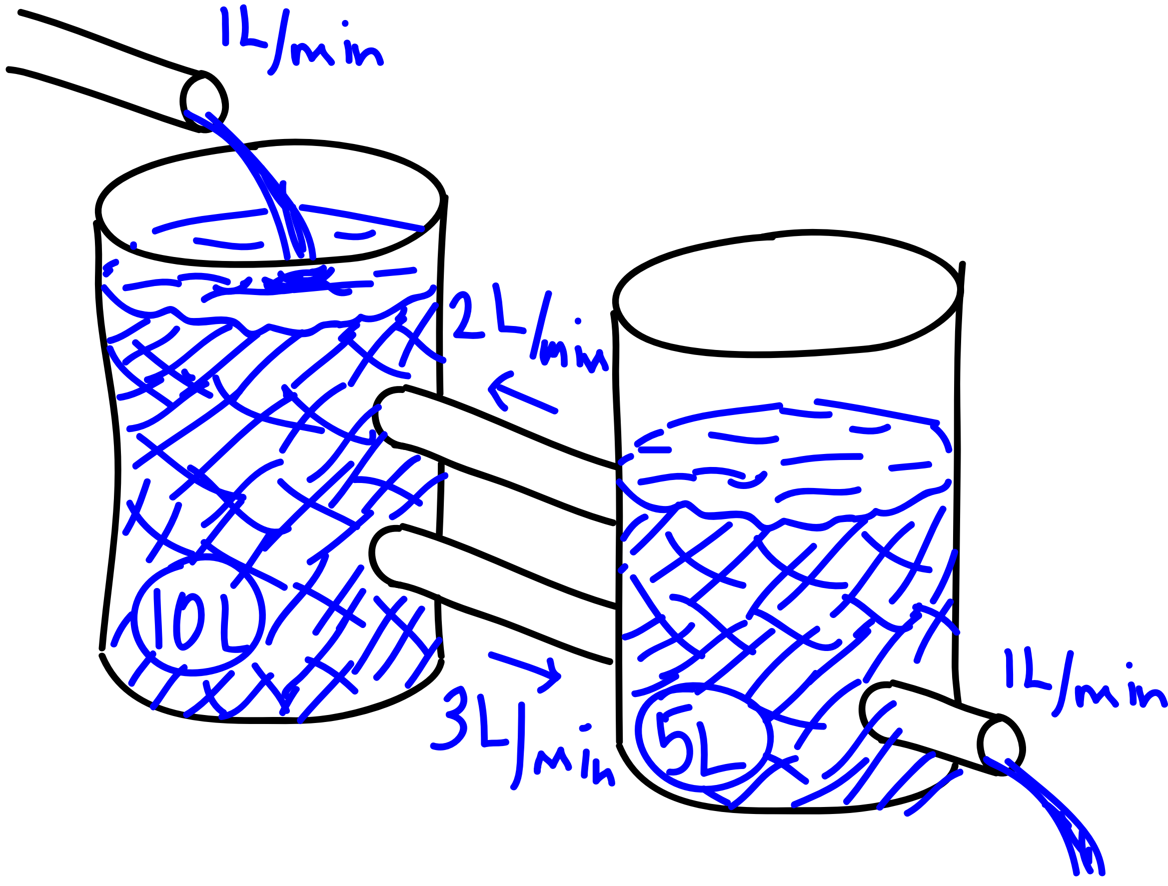
Nonhomogeneous system of DEs - example

- Salt water flows into a tank holding 10 L of water at a rate of 1 L/min with a concentration of 200 g/L. The well-mixed solution flows from that tank into a tank holding 5 L through a pipe at 3 L/min. Another pipe takes the solution in the second tank back into the first at a rate of 2 L/min. Finally, solution drains out of the second tank at a rate of 1 L/min.
- Write down a system of equations in matrix form for the mass of salt in each tank.



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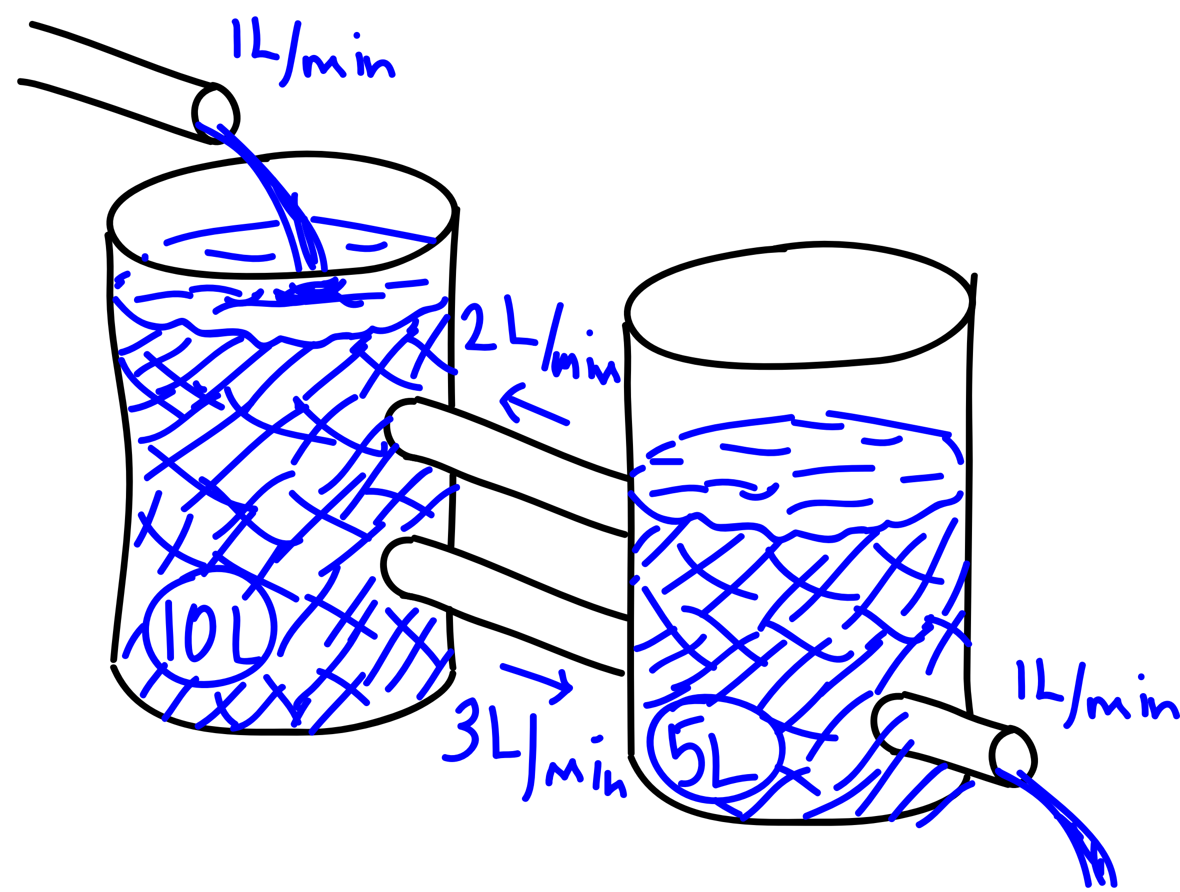

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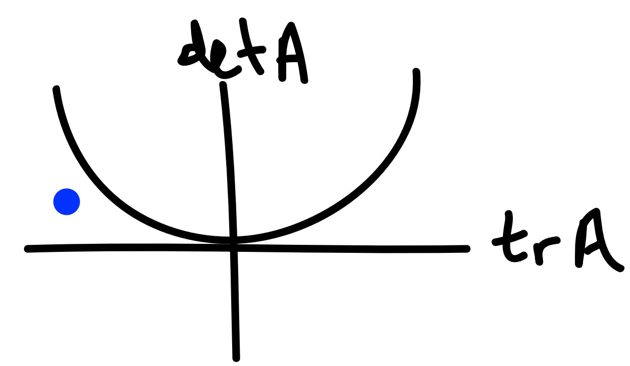
$$\text{tr}A = -\frac{9}{10} \qquad (\text{tr}A)^2 = \frac{81}{100}$$

$$\det A = \frac{9}{50} - \frac{6}{50} = \frac{3}{50} \qquad 4\det A = \frac{12}{50}$$



Both evalues negative!

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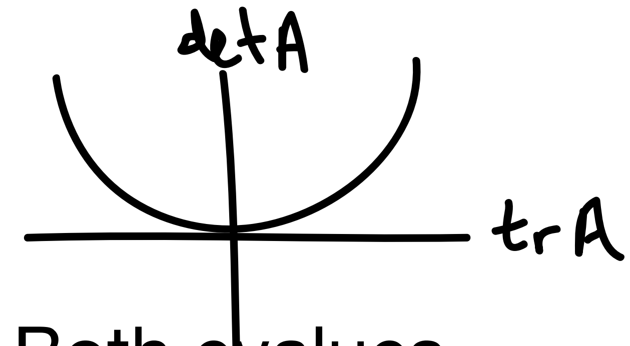

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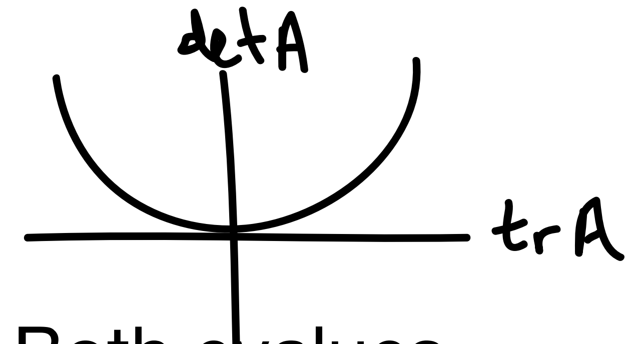

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$$\mathbf{m_h}(t) = C_1 e^{\lambda_1 t}\mathbf{v_1} + C_2 e^{\lambda_2 t}\mathbf{v_2} \qquad \left(\lambda_{1,2} = -\frac{9}{20} \pm \frac{\sqrt{57}}{20}\right)$$

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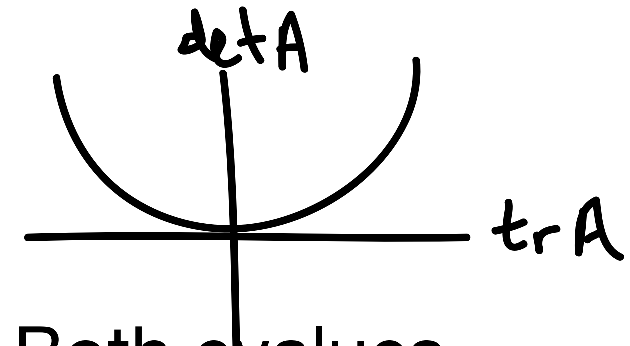

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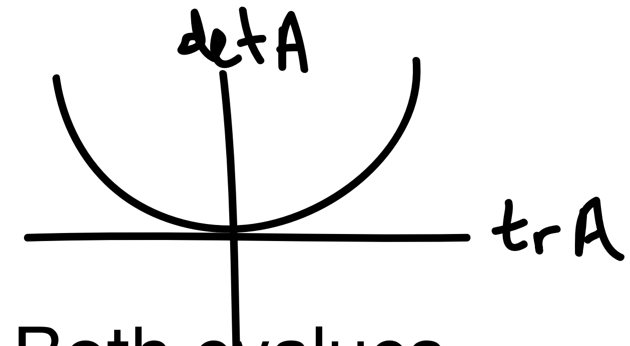

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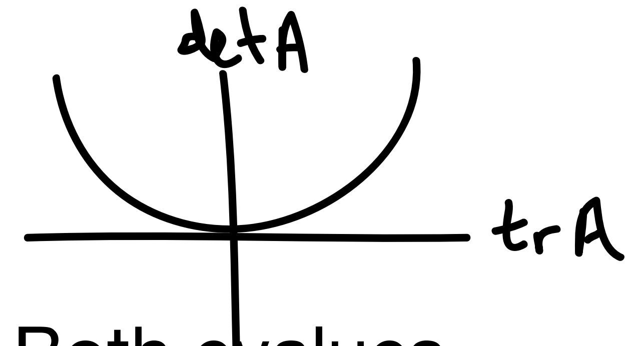

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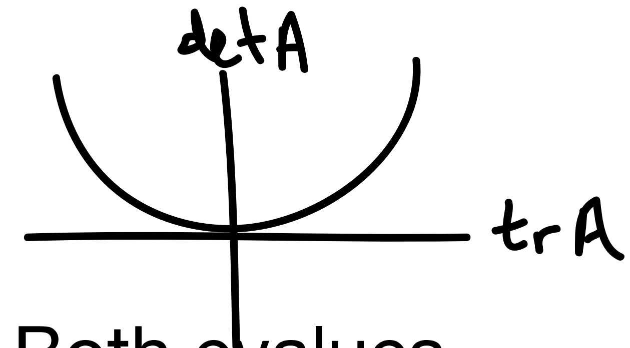

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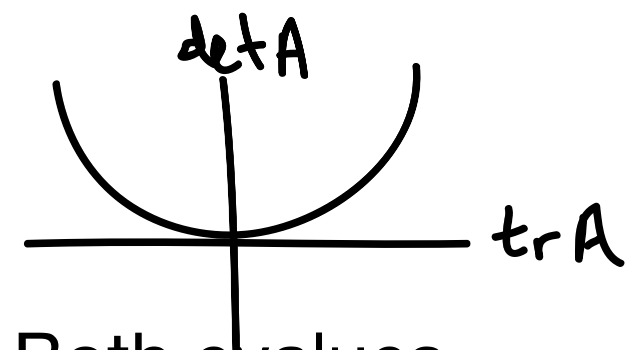

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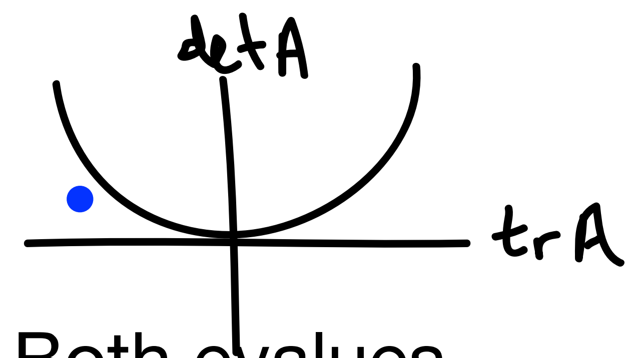

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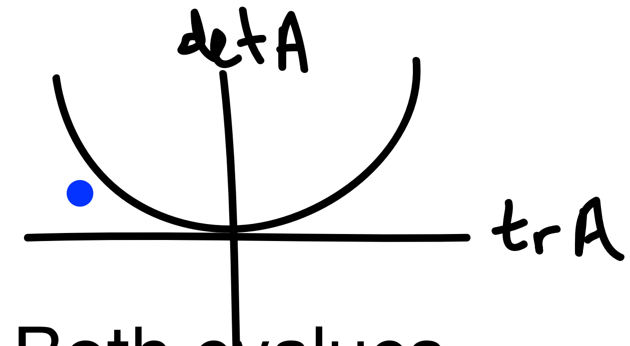

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Nonhomogeneous case - example

- A “Method of undetermined coefficients” similar to what we saw for second order equations can be used for systems.

- For this course, I’ll only test you on constant nonhomogeneous terms (like the previous example).

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$$g(t) = \begin{cases} \sin(\omega t) & 0 < t < 10, \\ 0 & t \geq 10. \end{cases}$$

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  - These can be handled by previous techniques (modified) but it isn't pretty (solve from t=0 to t=10, use y(10) as the IC for a new problem starting at t=10).

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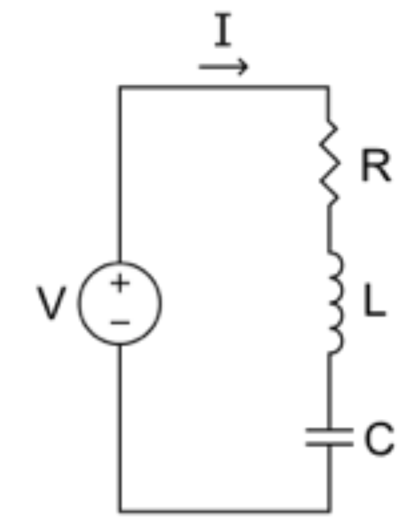

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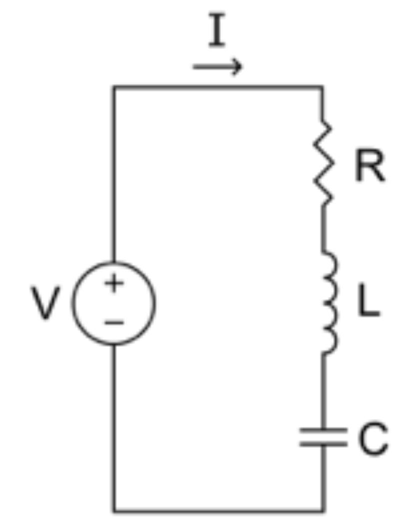

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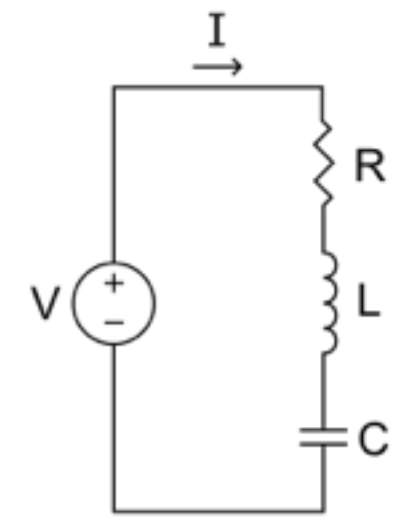

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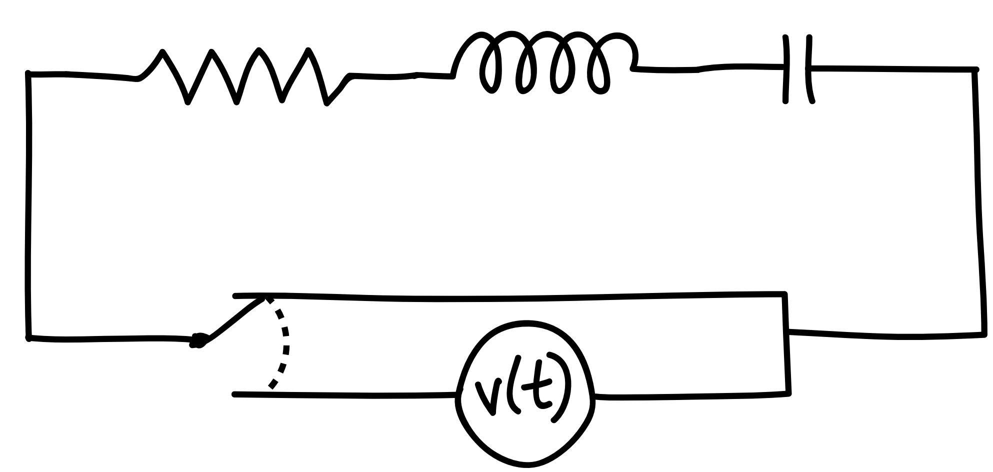
  - If v(t) comes from radio waves then $v(t) = A\cos(\omega t)$ and the circuit is called a radio receiver.
  - For $v(t) = \begin{cases} 1 & 0 < t < 10 \\ 0 & t \geq 10 \end{cases}$, the circuit has a switch that gets flipped at t=10.



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- Idea:

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Unknown y(t) that satisfies some ODE → (solve ODE) → Found y(t)

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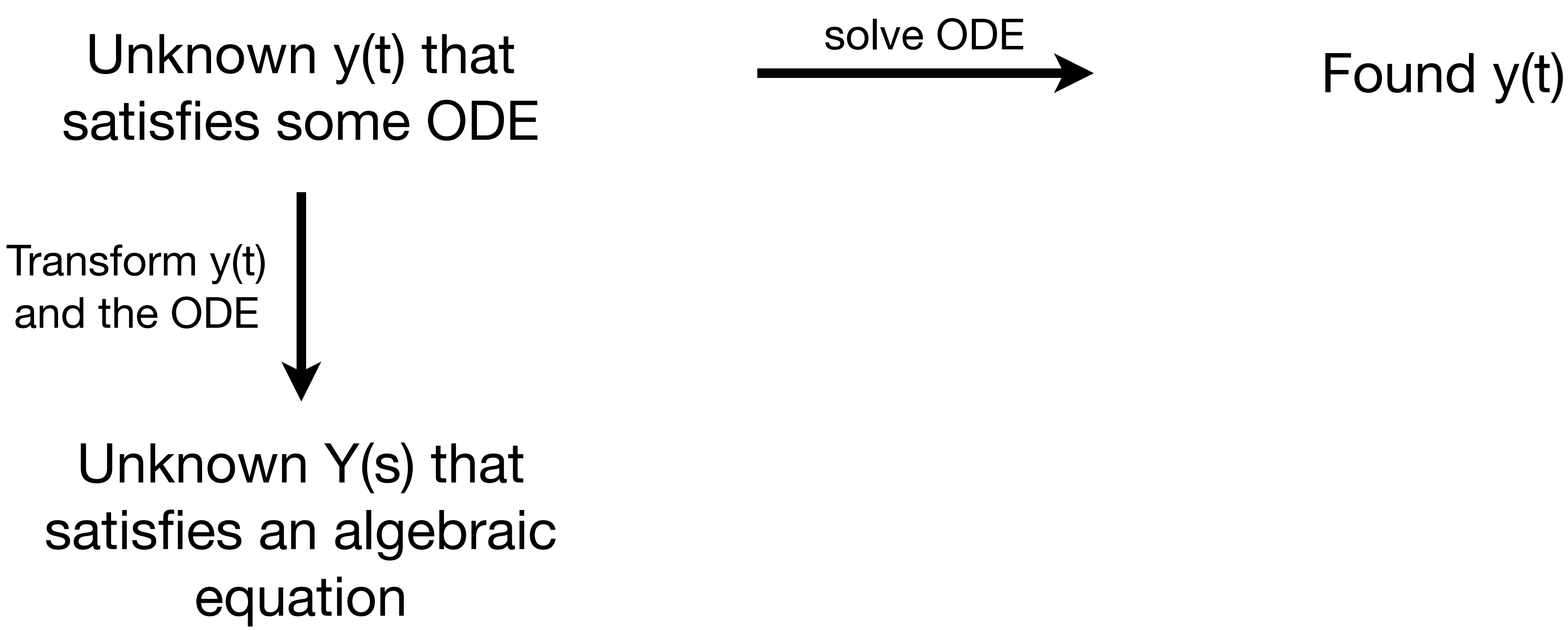

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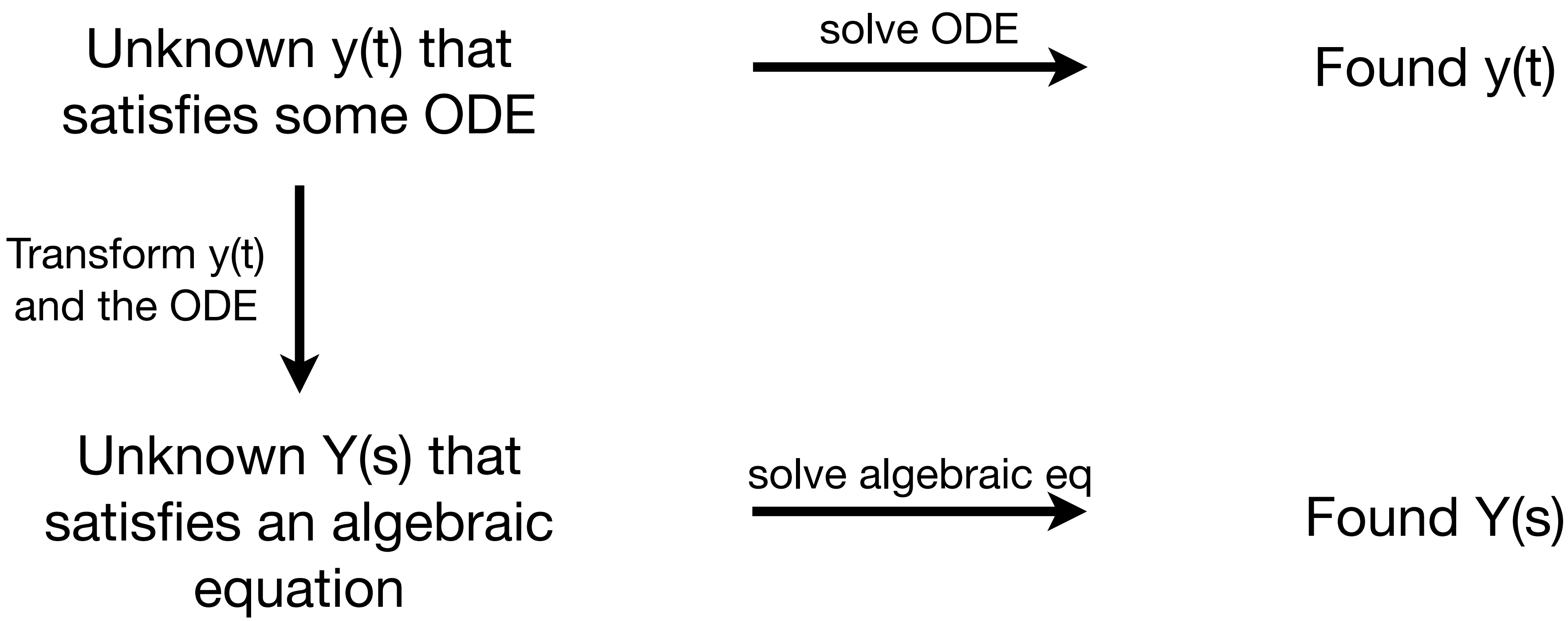

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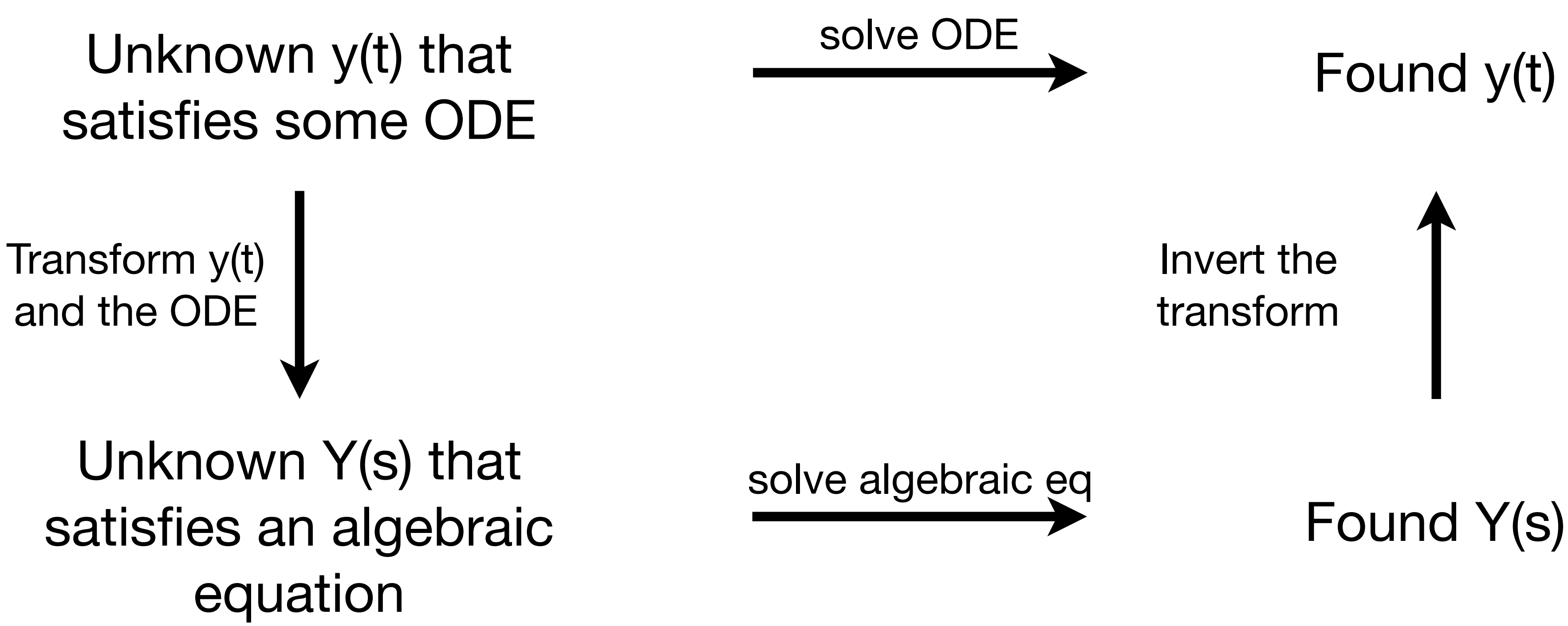

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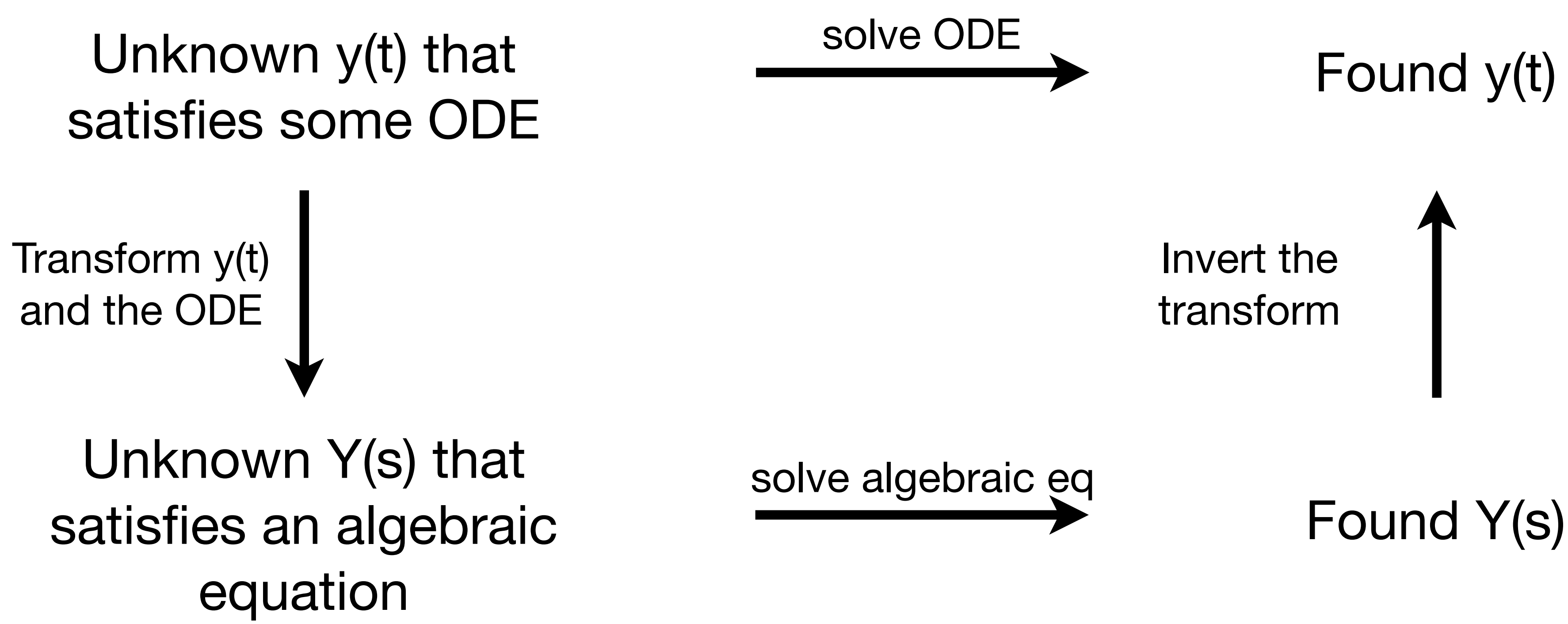

- Laplace transform of y(t): $\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} y(t)\ dt$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $y(t) = 3$?

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$$= -\frac{3}{s} e^{-st} \Big|_0^\infty$$

$$= \lim_{A \to \infty} -\frac{3}{s} e^{-st} \Big|_0^A$$

$$= -\frac{3}{s} \left(\lim_{A \to \infty} e^{-sA} - 1 \right)$$

$$= \frac{3}{s}$$ provided $s > 0$ and does not exist otherwise.

Laplace transforms - examples (6.1)

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$$= \frac{3}{s}$$

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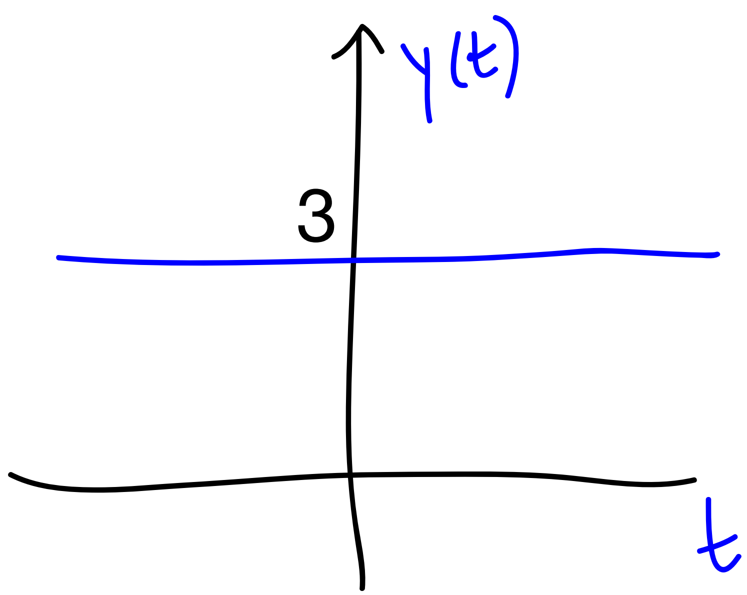

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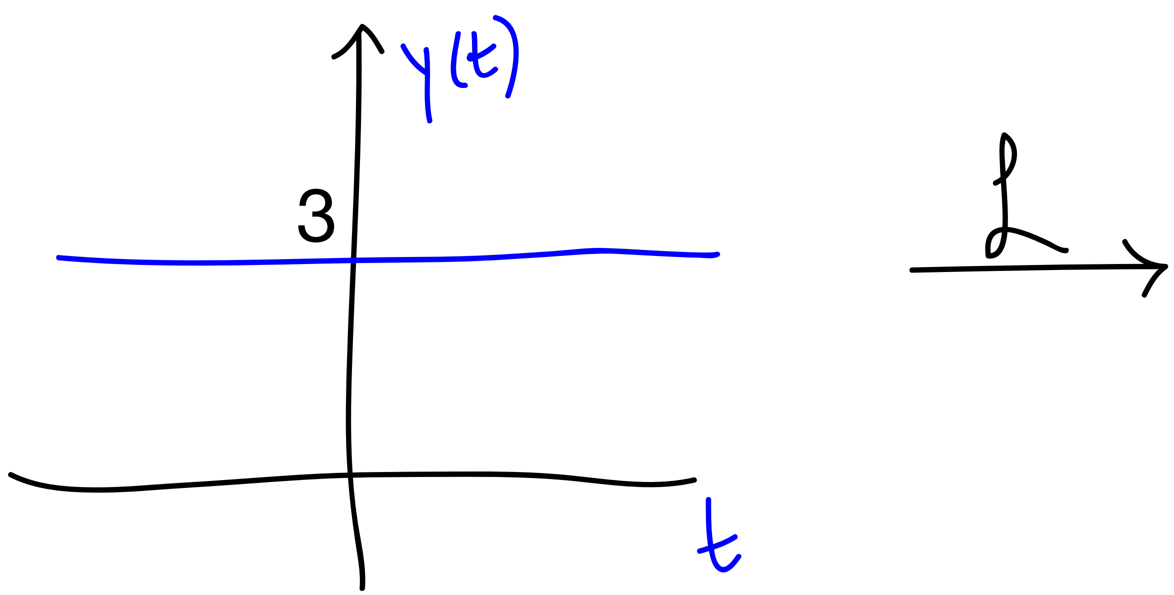

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