Today

- Reminders:
    - Pre-lecture assignment for Thursday 7 am
    - Week 1 assignment due Friday 5 pm.
- Finish up separating variables
- Modeling
- Existence and uniqueness (not going to test on the theory)

Separable equations (Section 2.2)

- What is $\frac{d}{dt}e^y$?

Hint: rewrite as $e^y \frac{dy}{dt} = t^2$.

$$\frac{d}{dt}(e^y) = t^2$$

$$e^y = \frac{1}{3}t^3 + C$$

(D) ye … dt

(E) Don't know.

- Solve $\frac{dy}{dt} = e^{-y}t^2$.

(A) $y(t) = t^2 e^t + C$

(B) $y(t) = \frac{1}{3}t^3 + C$

(C) $y(t) = \ln\left(\frac{1}{3}t^3\right) + C$

(D) $y(t) = \ln\left(\frac{1}{3}t^3 + C\right)$

(E) Don't know.

Separable equations (Section 2.2)

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Hint: rewrite as $e^{y}\dfrac{dy}{dt} = t^2$.

$$\frac{d}{dt}(e^{y}) = t^2$$

$$e^{y} = \frac{1}{3}t^3 + C$$

(D) $ye^{y}\ \dfrac{\quad}{dt}$

(E) Don't know.

- Solve $\dfrac{dy}{dt} = e^{-y}t^2$.

(A) $y(t) = t^2 e^t + C$

(B) $y(t) = \dfrac{1}{3}t^3 + C$

(C) $y(t) = \ln\left(\dfrac{1}{3}t^3\right) + C$

(D) $y(t) = \ln\left(\dfrac{1}{3}t^3 + C\right)$

(E) Don't know.

Separable equations (Section 2.2)

- First order ODEs of the form: $\frac{dy}{dx} = f(x)h(y)$
- Rename h(y) = 1/g(y): $\frac{dy}{dx} = \frac{f(x)}{g(y)}$
- Rewrite as $g(y)\frac{dy}{dx} = f(x)$.
- Rewrite g and f as derivatives of other functions: $G'(y)\frac{dy}{dx} = F'(x)$.
- Recognize a chain rule: $\frac{d}{dx}(G(y)) = G'(y)\frac{dy}{dx}$.
- Take antiderivatives to get $G(y) = F(x) + C$.
- Finally, solve for y if possible: $y(x) = G^{-1}(F(x) + C)$.

Separable equations (Section 2.2)

- Solve: $$\frac{dy}{dx} = -\frac{x}{y}$$

(A) $y(x) = x$

(B) $y(x) = \sqrt{C - x^2}$

(C) $y(x) = \sqrt{x^2 + C}$

(D) $y(x) = C - x^2$

(E) None of these (or don't know)

$$y\frac{dy}{dx} = -x$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + D$$

$$y^2 = -x^2 + C$$

Does (B) cover all possible initial conditions?

Separable equations (Section 2.2)

$$y(x) = \sqrt{C - x^2}$$

- y(0)=2 ----> C=4
- y(1)=1 ----> C=2
- y(1)=-2 ----> C=?
- General solution: $y = \pm\sqrt{C - x^2}$
- Or express implicitly: $y^2 = -x^2 + C$
- To satisfy an IC, must choose a value for C and + or - .

Separable equations (Section 2.2)

- Solve: $$\frac{dy}{dt} = \frac{1}{\cos(y)}$$

(A) $y(t) = \sin(t)$

(B) $y(t) = \arcsin(t + C)$

(C) $\sin(y) = t + C$

(D) $y(t) = \arcsin(t) + C$

(E) $y(t) = \arccos(t + C)$

Separable equations (Section 2.2)

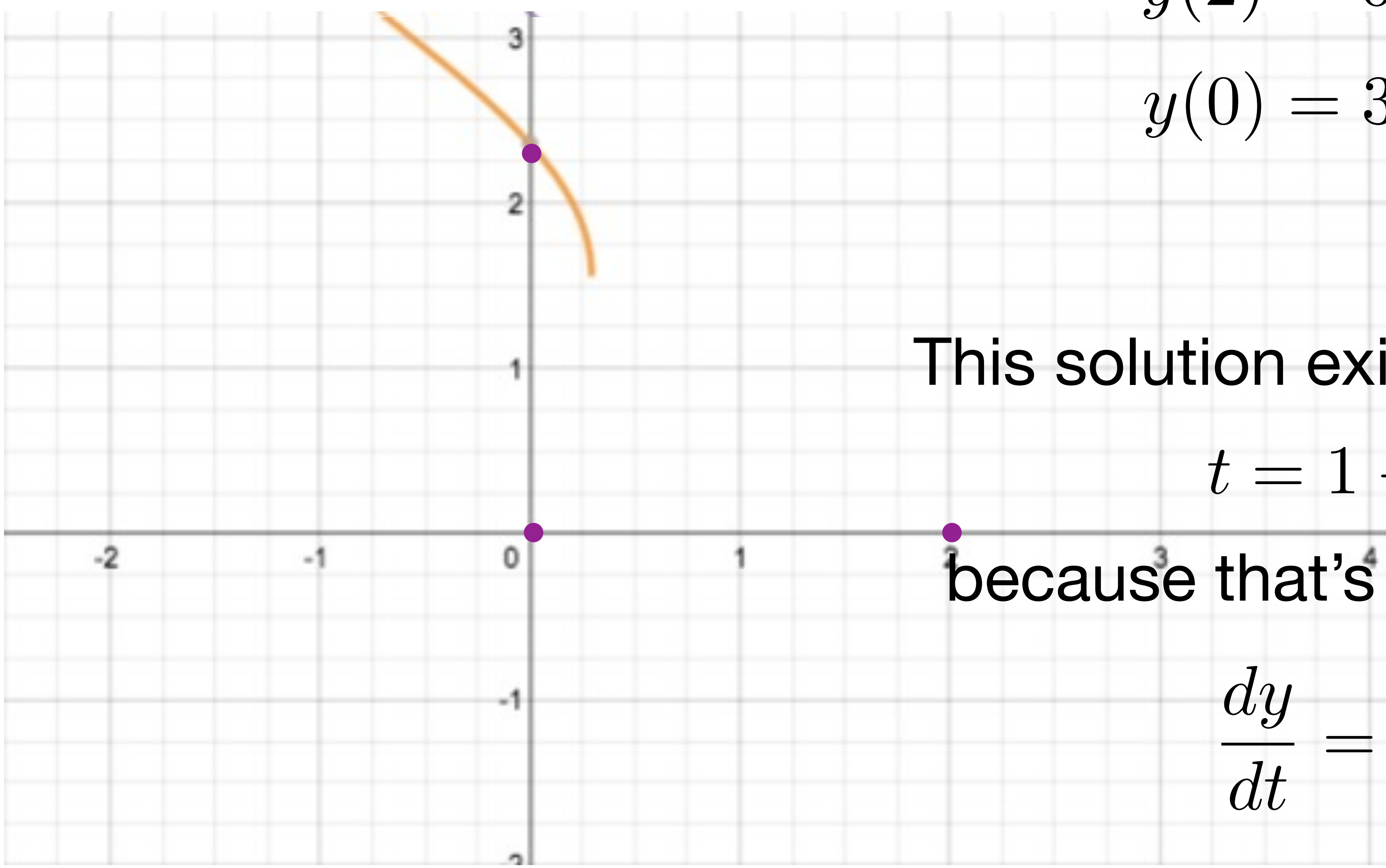
$\sin(y) = t + C$

$y(t) = \arcsin(t + C)$ with IC

$y(0) = 0 \qquad C = 0$

$y(2) = 0 \qquad C = -2$

$y(0) = 3\pi/4 \quad C = \dfrac{1}{\sqrt{2}}$



This solution exists only up until

$$t = 1 - 1/\sqrt{2}$$

because that's when y = π/2.

$$\frac{dy}{dt} = \frac{1}{\cos(y)}$$

Modeling (Section 2.3)

- Inflow/outflow problems
    - Determine what quantity(-ies) to track (e.g. mass, concentration, temperature, etc.).
    - Choose a small interval of time, Δt, and add up all the changes.
    - Note that $q(t + \Delta t) = q(t) +$ change during intervening Δt .
    - Take limit as $\Delta t \to 0$ to get an equation for q(t).

Modeling (Section 2.3) - Example

- Freshwater flows into a tank at a rate 2 L/min. The tank starts with a concentration of 100 g / L of salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank ($\lim_{t \to \infty} m(t)$)?

(a) What is the change in the mass of salt in any short interval of time Δt?

(A) $\Delta m \approx -2 \text{ L/min} \times m(t) \;/\; 10 \text{ L}$

(B) $\Delta m \approx -2 \text{ L/min} \times 100 \text{ g/L} \times \Delta t$

★ (C) $\Delta m \approx -2 \text{ L/min} \times m(t) \;/\; 10 \text{ L} \times \Delta t$

(D) $\Delta m \approx -2 \text{ L/min} \times m(t)$

Modeling (Section 2.3) - Example

- Freshwater flows into a tank at a rate 2 L/min. The tank starts with a concentration of 100 g / L of salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank ($\lim_{t\to\infty} m(t)$)?

- $m(t+\Delta t) = m(t) + \Delta m$ so
- $m(t+\Delta t) \approx m(t) - \Delta t \ \times \ 2 \text{ L/min} \ \times \ m(t) \ / \ 10 \text{ L}$
- Rearranging: $\dfrac{m(t+\Delta t) - m(t)}{\Delta t} \approx -\dfrac{1}{5}m(t)$
- Finally, taking a limit:

$$\boxed{\frac{dm}{dt} = -\frac{1}{5}m(t)}$$

Modeling (Section 2.3) - Example

- Freshwater flows into a tank at a rate 2 L/min. The tank starts with a concentration of 100 g / L of salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank ($\lim_{t\to\infty} m(t)$)?

(a) We got the equation (m'=-1/5 m). Now what is the initial condition?

- m(0) = 1000 g.

Modeling (Section 2.3) - Example

- Freshwater flows into a tank at a rate 2 L/min. The tank starts with a concentration of 100 g / L of salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank ($\lim_{t\to\infty} m(t)$)?

- What method could you use to solve the ODE $\dfrac{dm}{dt} = -\dfrac{1}{5}m(t)$?

  (A) Integrating factors.

  (B) Separating variables.

  (C) Just knowing some derivatives.

  ☆ (D) All of these.

  (E) None of these.

To think about: what is the most general equation that can be solved using (A) and (B)?

Modeling (Section 2.3) - Example

- Freshwater flows into a tank at a rate 2 L/min. The tank starts with a concentration of 100 g / L of salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank ($\lim_{t \to \infty} m(t)$)?

- The solution to the IVP is

  (A) $m(t) = Ce^{-t/5}$

  (B) $m(t) = 100e^{-t/5}$

  (C) $m(t) = 100e^{t/5}$

  (D) $m(t) = 1000e^{t/5}$

  ★ (E) $m(t) = 1000e^{-t/5}$

Answer to (b)?

$$\lim_{t \to \infty} m(t) = 0$$

Modeling (Section 2.3) - Example

- Saltwater with a concentration of 200 g/L flows into a tank at a rate 2 L/min. The tank starts with no salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank?

(a) The IVP is

(A) $m' = 200 - 2m, \quad m(0) = 0$

(B) $m' = 400 - 2m, \quad m(0) = 200$

★ (C) $m' = 400 - m/5, \quad m(0) = 0$

(D) $m' = 200 - m/5, \quad m(0) = 0$

(E) $m' = 400 - m/5, \quad m(0) = 200$

Modeling (Section 2.3) - Example

- Saltwater with a concentration of 200 g/L flows into a tank at a rate 2 L/min. The tank starts with no salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank?

(b) Directly from the equation ($m' = 400 - m/5$), find an m for which $m'=0$.

- $m=2000$. Called steady state - a constant solution.
- What happens when $m < 2000$? ---> $m' > 0$.
- What happens when $m > 2000$? ---> $m' < 0$.
- Limiting mass: 2000 g (Long way: solve the eq. and let $t \to \infty$.)

Existence and uniqueness (Section 2.4)

Theorem 2.4.2 Let the functions f and $\dfrac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta, \quad \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t_0 < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the IVP

$$y' = f(t, y), \quad y(t_0) = y_0.$$

- A couple questions/examples to explore on your own:
  - Why don't we get a solution all the way to the ends of the t interval?
    - Example: $\dfrac{dy}{dt} = y^2, \quad y(0) = 1$
  - How does a non-continuous RHS lead to more than one solution?
    - Example: $\dfrac{dy}{dt} = \sqrt{y}, \quad y(0) = 0$

Second order linear equations (Chapter 3)

- The general form for a second order linear equation:

$$y'' + p(t)y' + q(t)y = g(t)$$

- Now, an IVP requires two ICs:

$$y(0) = y_0, \quad y'(0) = v_0$$

- As with first order linear equations, we have homogeneous (g=0) and non-homogeneous second order linear equations.

- We'll start by considering the homogeneous case with constant coefficients:

$$ay'' + by' + cy = 0$$

Homog. eq. with constant coeff. (Section 3.1)

$$ay'' + by' + cy = 0$$

- Suppose you already found a couple solutions, $y_1(t)$ and $y_2(t)$. This means that

$$ay_1'' + by_1' + cy_1 = 0 \qquad \text{and} \qquad ay_2'' + by_2' + cy_2 = 0$$

- Notice that $y(t) = C_1y_1(t)$ is also a solution. Plug it in and check:

$$\begin{aligned} &a(C_1y_1)'' + b(C_1y_1)' + c(C_1y_1) \\ &\quad = aC_1(y_1)'' + bC_1(y_1)' + cC_1(y_1) \\ &= C_1(ay_1'' + by_1' + cy_1) = 0 \end{aligned}$$

Homog. eq. with constant coeff. (Section 3.1)

- Which of the following functions are also solutions?

  (A) $y(t) = y_1(t)^2$

  (B) $y(t) = y_1(t)+y_2(t)$

  (C) $y(t) = y_1(t)\, y_2(t)$

  (D) $y(t) = y_1(t) / y_2(t)$

- In fact, the following are all solutions: $C_1y_1(t)$, $C_2y_2(t)$, $C_1y_1(t)+C_2y_2(t)$.
- With first order equations, the arbitrary constant appeared through an integration step in our methods. With second order equations, not so lucky.
- Instead, find two independent solutions, $y_1(t)$, $y_2(t)$, by whatever method.
- The general solution will be $y(t) = C_1y_1(t) + C_2y_2(t)$.