Repeated roots

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- There are three cases.

  i. Two distinct real roots: $b^2 - 4ac > 0$. ($r_1 \neq r_2$)

  ii. A repeated real root: $b^2 - 4ac = 0$.

  iii. Two complex roots: $b^2 - 4ac < 0$.

- For case ii ($r_1 = r_2 = r$), we need another independent solution!
- Reduction of order - a method for guessing another solution.

Reduction of order

- You have one solution $y_1(t)$ and you want to find another independent one, $y_2(t)$.
- Guess that $y_2(t) = v(t)y_1(t)$ for some as yet unknown $v(t)$. If you can find $v(t)$ this way, great. If not, gotta try something else.
- Example - $y'' + 4y' + 4y = 0.$ Only one root to the characteristic equation, r=-2, so we only get one solution that way: $y_1(t) = e^{-2t}$.
- Use Reduction of order to find a second solution.

$$y_2(t) = v(t)e^{-2t}$$

- Heuristic explanation for exponential solutions and Reduction of order.

Reduction of order

For the equation $y'' + 4y' + 4y = 0$, say you know $y_1(t) = e^{-2t}$.

Guess $y_2(t) = v(t)e^{-2t}$.

$$\Downarrow$$

$$\underline{4y_2(t) = 4\cancel{v(t)e^{-2t}}}$$

$$y_2'(t) = v'(t)e^{-2t} - 2v(t)e^{-2t}$$

$$\Downarrow$$

$$\underline{4y_2'(t) = \cancel{4v'(t)e^{-2t}} - 8\cancel{v(t)e^{-2t}}}$$

$$y_2''(t) = v''(t)e^{-2t} - 2v'(t)e^{-2t} - 2v'(t)e^{-2t} + 4v(t)e^{-2t}$$

$$\Searrow$$

$$\underline{y_2''(t) = \boxed{v''(t)e^{-2t}} - \cancel{4v'(t)e^{-2t}} + 4\cancel{v(t)e^{-2t}}}$$

$$0 = y_2'' + 4y_2' + 4y_2 = v''e^{-2t}$$

$$v'' = 0 \quad \Rightarrow \quad v' = C_1 \quad \Rightarrow \quad v(t) = C_1 t + C_2$$

Reduction of order

For the equation $y'' + 4y' + 4y = 0$, say you know $y_1(t) = e^{-2t}$.

Guess $y_2(t) = v(t)e^{-2t}$ (where $v(t) = C_1 t + C_2$).

$$= (C_1 t + C_2)e^{-2t}$$

$$y(t) = C_1 \underbrace{te^{-2t}}_{y_2(t)} + C_2 \underbrace{e^{-2t}}_{y_1(t)}$$

Is this the general solution? Calculate the Wronskian:

$$W(e^{-2t}, te^{-2t})(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^{-4t} \neq 0$$

So yes!

Summary

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- There are three cases.

i. Two distinct real roots: $b^2 - 4ac > 0$. (r_1, r_2)

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

ii.A repeated real root: $b^2 - 4ac = 0$. (r)

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

iii.Two complex roots: $b^2 - 4ac < 0$. ($r_{1,2} = \alpha \pm i\beta$)

$$y = e^{\alpha t} \left(C_1 \cos(\beta t) + C_2 \sin(\beta t) \right)$$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 8y = 0$$

(A) $y(t) = C_1 e^{-2t} + C_2 e^{-4t}$

★ (B) $y(t) = C_1 e^{2t} + C_2 e^{4t}$

(C) $y(t) = e^{2t}(C_1 \cos(4t) + C_2 \sin(4t))$

(D) $y(t) = e^{-2t}(C_1 \cos(4t) + C_2 \sin(4t))$

(E) $y(t) = C_1 e^{2t} + C_2 t e^{4t}$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 9y = 0$$

(A) $y(t) = C_1 e^{3t}$

(B) $y(t) = C_1 e^{3t} + C_2 e^{3t}$

(C) $y(t) = C_1 e^{3t} + C_2 e^{-3t}$

★ (D) $y(t) = C_1 e^{3t} + C_2 t e^{3t}$

(E) $y(t) = C_1 e^{3t} + C_2 v(t) e^{3t}$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 10y = 0$$

(A) $y(t) = C_1 e^{3t} + C_2 e^{t}$

(B) $y(t) = C_1 e^{3t} + C_2 e^{-t}$

(C) $y(t) = C_1 \cos(3t) + C_2 \sin(3t)$

(D) $y(t) = e^{t}(C_1 \cos(3t) + C_2 \sin(3t))$

★ (E) $y(t) = e^{3t}(C_1 \cos(t) + C_2 \sin(t))$

Second order, linear, constant coeff, **non**homogeneous (3.5)

- Our next goal is to figure out how to find solutions to nonhomogeneous equations like this one:

$$y'' - 6y' + 8y = \sin(2t)$$

- But first, a bit more on the connections between matrix algebra and differential equations . . .

Some connections to linear (matrix) algebra

- An mxn matrix is a gizmo that takes an n-vector and returns an m-vector:

$$\overline{y} = A\overline{x}$$

- It is called a linear operator because it has the following properties:

$$A(c\overline{x}) = cA\overline{x}$$
$$A(\overline{x} + \overline{y}) = A\overline{x} + A\overline{y}$$

- Not all operators work on vectors. Derivative operators take a function and return a new function. For example,

$$z = L[y] = \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y$$

- This one is linear because

$$L[cy] = cL[y]$$
$$L[y + z] = L[y] + L[z]$$

Note: y, z are functions of t and c is a constant.

Some connections to linear (matrix) algebra

- A homogeneous matrix equation has the form

$$A\overline{x} = \overline{0}$$

- A non-homogeneous matrix equation has the form

$$A\overline{x} = \overline{b}$$

- A homogeneous differential equation has the form

$$L[y] = 0$$

- A non-homogeneous differential equation has the form

$$L[y] = g(t)$$

Solutions to homogeneous matrix equations

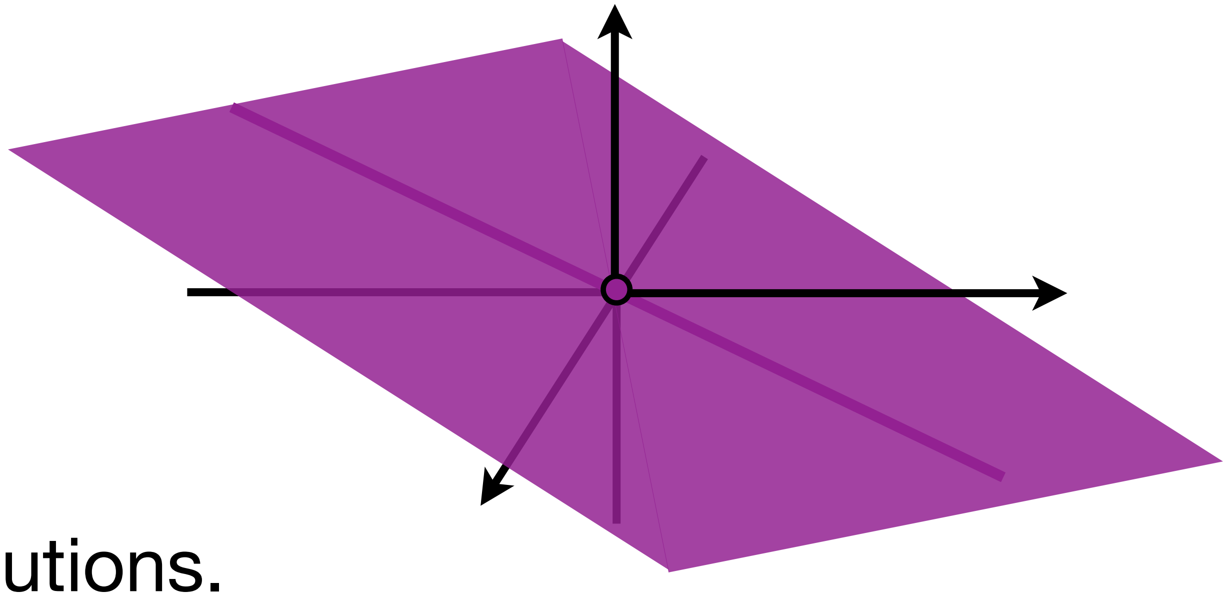
- The matrix equation $A\overline{x} = \overline{0}$ could have (depending on A)

☆ (A) no solutions.

⇨ (B) exactly one solution.

⇨ (C) a one-parameter family of solutions.

⇨ (D) an n-parameter family of solutions.



Possibilities:

$$\overline{x} = \overline{0}$$

$$\overline{x} = C\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\overline{x} = C_1\begin{pmatrix}1\\-1\\1\end{pmatrix} + C_2\begin{pmatrix}0\\2\\1\end{pmatrix}$$

Choose the answer that is incorrect.

Solutions to homogeneous matrix equations

- Example 1. Solve the equation $A\overline{x} = \overline{0}$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & -2 \\ 2 & 1 & 1 \end{pmatrix}$$

Each equation describes a plane.

- Row reduction gives

$$A \sim \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{pmatrix}$$

In this case, only two of them really matter.

- so $x_1 - \dfrac{1}{3}x_3 = 0$ and $x_2 + \dfrac{5}{3}x_3 = 0$ and x_3 can be whatever (because it doesn't have a leading one).

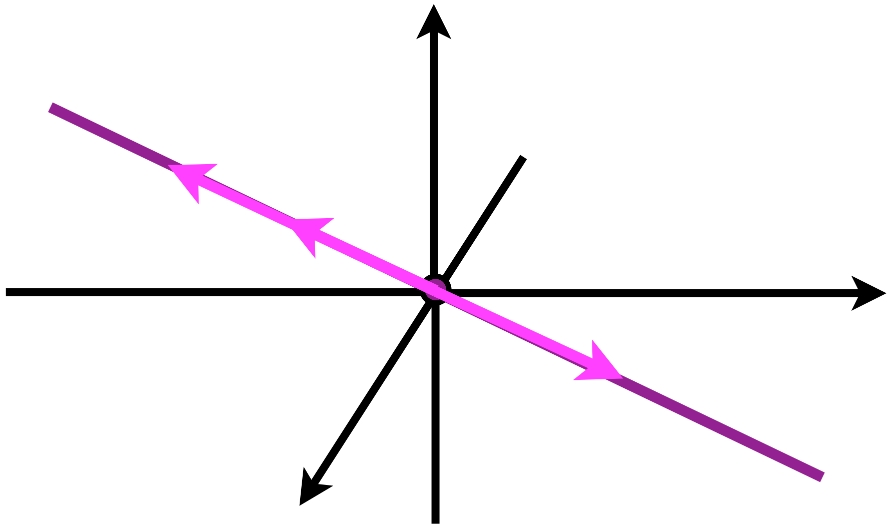
Solutions to homogeneous matrix equations

- Example 1. Solve the equation $A\overline{x} = \overline{0}$.
- so $x_1 - \frac{1}{3}x_3 = 0$ and $x_2 + \frac{5}{3}x_3 = 0$ and x_3 can be whatever.

$$x_1 = \frac{1}{3}x_3 \qquad x_1 = \frac{1}{3}C$$

$$x_2 = -\frac{5}{3}x_3 \qquad x_2 = -\frac{5}{3}C$$

$$x_3 = C$$



- Thus, the solution can be written as $\overline{x} = \frac{C}{3}\,C'\begin{pmatrix} \mathbf{1} \\ \mathbf{-5} \\ \mathbf{3} \end{pmatrix}$.

Solutions to homogeneous matrix equations

- Example 2. Solve the equation $A\overline{x} = \overline{0}$ where

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

- Row reduction gives

$$A \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- so $x_1 - 2x_2 + x_3 = 0$ and both x_2 and x_3 can be whatever.

$$\overline{x} = C_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

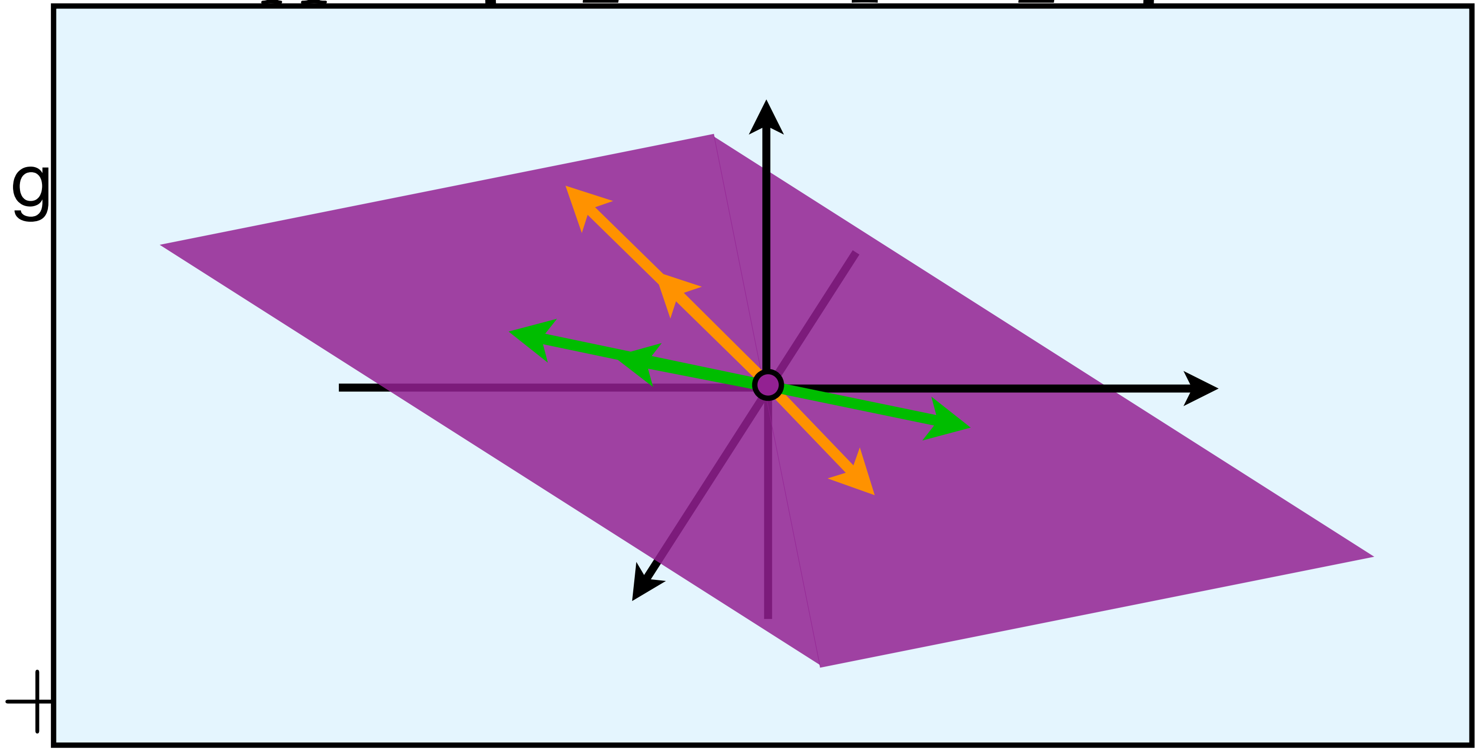
Solutions to homogeneous matrix equations

- Example 2. Solve the equation $A\overline{x} = \overline{0}$ where

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \end{pmatrix}$$

- Row reduction g

- so $x_1 - 2x_2 +$ atever.



$$\overline{x} = C_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Solutions to non-homogeneous matrix equations

- Example 3. Solve the equation $A\overline{x} = \overline{b}$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & -2 \\ 2 & 1 & 1 \end{pmatrix} \text{ and } \overline{b} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

- Row reduction gives

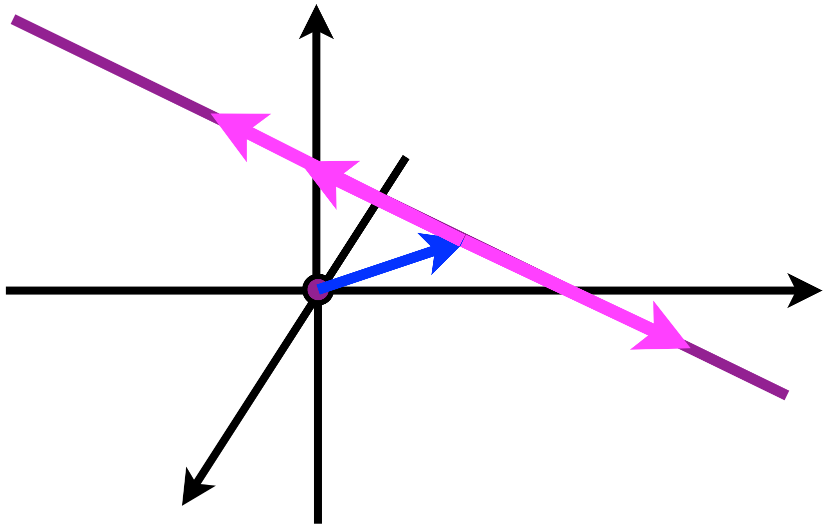
$$\left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 5/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- so $x_1 - \frac{1}{3}x_3 = \frac{2}{3}$ and $x_2 + \frac{5}{3}x_3 = \frac{2}{3}$ and x_3 can be whatever.

Solutions to non-homogeneous matrix equations

- Example 3. Solve the equation $A\overline{x} = \overline{b}$.
- so $x_1 - \frac{1}{3}x_3 = \frac{2}{3}$ and $x_2 + \frac{5}{3}x_3 = \frac{2}{3}$ and x_3 can be whatever.

$$x_1 = \frac{1}{3}x_3 + \frac{2}{3} \qquad x_2 = -\frac{5}{3}x_3 + \frac{2}{3}$$



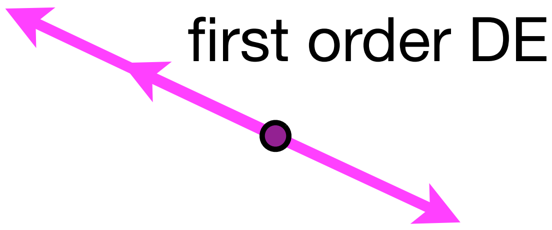
$$\overline{\boldsymbol{x}} = C'\frac{C}{3}\begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 2/3 \\ 0 \end{pmatrix}$$

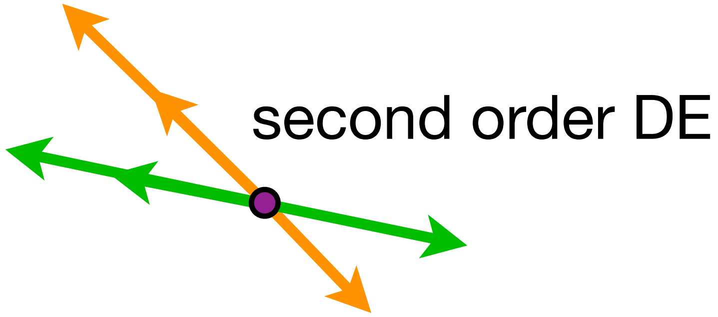
the general solution to the homogeneous problem

one particular solution to nonhomogeneous problem

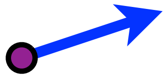
Solutions to nonhomogeneous differential equations

- To solve a nonhomogeneous differential equation:
  1. Find the general solution to the associated homogeneous problem, $y_h(t)$.



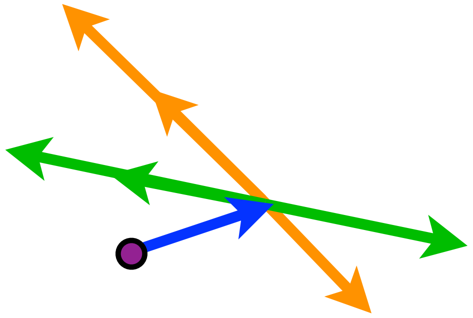


  2. Find a particular solution to the nonhomogeneous problem, $y_p(t)$.



  3. The general solution to the nonhomogeneous problem is their sum:

$$y = y_h + y_p = \underbrace{C_1 y_1}_{} + \underbrace{C_2 y_2}_{} + \underbrace{y_p}_{}$$



- For step 2, try "Method of undetermined coefficients"...