Today

- Step and ramp functions (continued)
- The Dirac Delta function and impulse force
- (Modeling with delta-function forcing)

Step function forcing

- Solve using Laplace transforms:

$$y'' + 2y' + 10y = g(t) = \begin{cases} 0 & \text{for } t < 2 \text{ and } t \geq 5, \\ 1 & \text{for } 2 \leq t < 5. \end{cases}$$

$$y(0) = 0, \;\; y'(0) = 0.$$

- The transformed equation is

$$s^2 Y(s) + 2sY(s) + 10Y(s) = \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s}.$$

$$Y(s) = \frac{e^{-2s} - e^{-5s}}{s(s^2 + 2s + 10)} = (e^{-2s} - e^{-5s})H(s).$$

$$H(s) = \frac{1}{s(s^2 + 2s + 10)}$$

- Recall that $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-sc}F(s)$

$$y(t) = u_2(t)h(t-2) - u_5(t)h(t-5)$$

- So we just need h(t) and we're done.

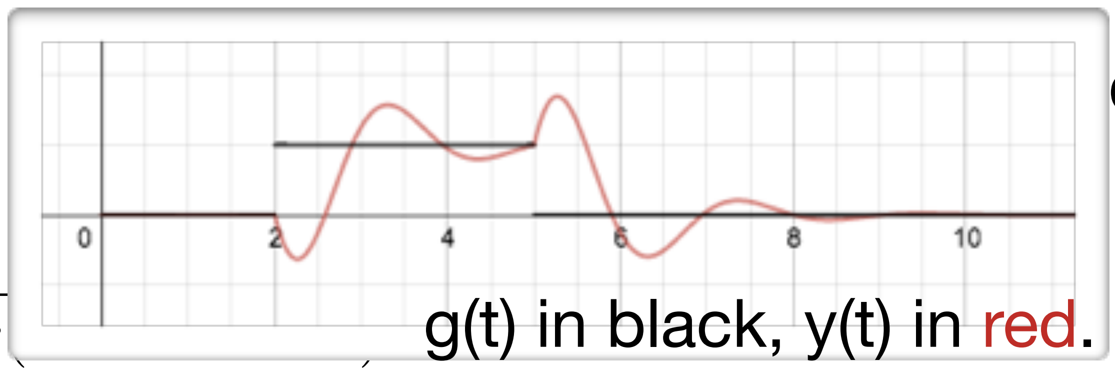
Step function forcing

- Inverting H(s) to get h(t): $H(s) = \dfrac{1}{s(s^2+2s+10)}$

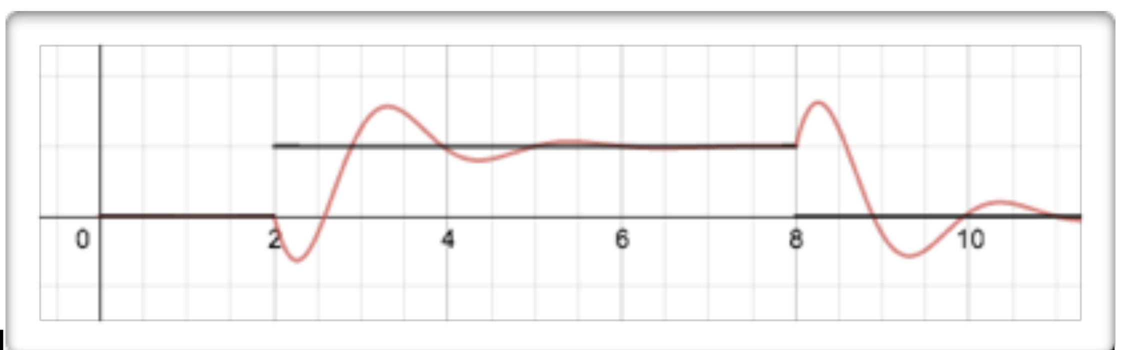
Partial fraction decomposition!

•Does ... ors.

$H(s) = \dfrac{\ }{s}$ ...



g(t) in black, y(t) in red.



$$y(t) = u_2(t)h(t-2) - u_5(t)h(t-5)$$

- See Su... calculation (pdf and video): https://wiki.math.ubc.ca/mathbook/M256/Resources

$$\boxed{h(t) = \frac{1}{10} - \frac{1}{10}e^{-t}\cos(3t) - \frac{1}{30}\cdot e^{-t}\sin 3t}$$

Step function forcing

- An example with a ramped forcing function:

Two methods:

1. Build from left to right, adding/subtracting what you need to make the next section:

$$g(t) = u_5(t)\frac{1}{5}(t-5) - u_{10}(t)\frac{1}{5}(t-10)$$

2. Build each section independently:

$$g(t) = \left(u_5(t) - u_{10}(t)\right)\frac{1}{5}(t-5) + u_{10}(t)\cdot 1$$

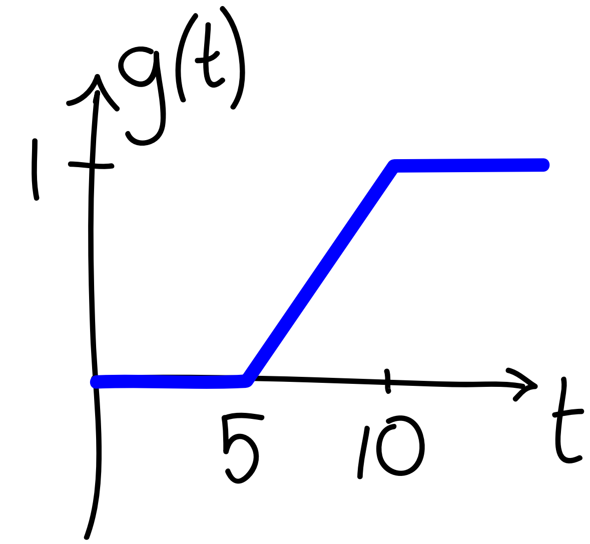
- W

★ (C) $g(t) = (u_5(t)(t-5) - u_{10}(t)(t-10))/5$

(D) $g(t) = (u_5(t)(t-5) - u_{10}(t)(t-10))/10$

Step function forcing

- An example with a ramped forcing function:



$$y'' + 4y = u_5(t)\frac{1}{5}(t-5) - u_{10}(t)\frac{1}{5}(t-10)$$

$$y(0) = 0, \;\; y'(0) = 0.$$

$$s^2 Y + 4Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}$$

$$Y(s) = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2(s^2+4)} \quad = \frac{1}{5}(e^{-5s} - e^{-10s})H(s)$$

$$y(t) = \frac{1}{5}[u_5(t)h(t-5) - u_{10}(t)h(t-10)]$$

Find h(t) given that $H(s) = \dfrac{1}{s^2(s^2+4)}$.

$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

Delta-function forcing

- Suppose a mass is sitting at position x and a force g(t) acts on it:

$$mx'' = g(t)$$

- To find x(t), integrate up:

$$\int_a^b mx''\ dt = \int_a^b g(t)\ dt$$

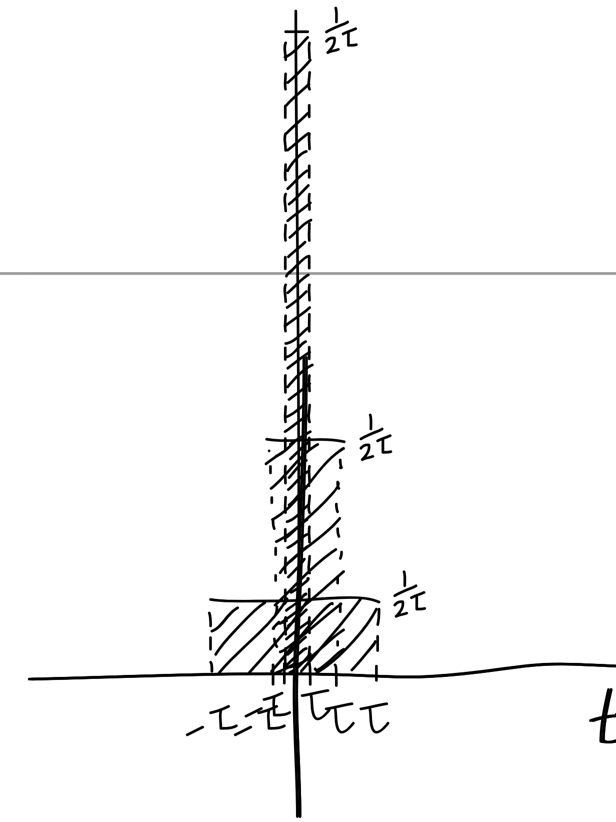
$$mx'\Big|_a^b = \int_a^b g(t)\ dt$$

$$mv(b) - mv(a) = \int_a^b g(t)\ dt$$

- $\int_a^b g(t)\ dt$ is the change in momentum of the mass - called impulse.

- If the force is large and sudden (say a hammer hitting the mass), maybe we just need to get this integral correct and the details don't matter.

Delta-function forcing



- Let's assume $g(t) = \begin{cases} \dfrac{I_0}{2\tau} & -\tau < t < \tau \\ & \\ 0 & \text{otherwise} \end{cases}$

$$= (u_{-\tau}(t) - u_\tau(t))\frac{I_0}{2\tau}$$

impulse = $\Delta\text{momentum} = \displaystyle\int_{-\infty}^{\infty} g(t)\ dt = \int_{-\tau}^{\tau} \frac{I_0}{2\tau}\ dt = I_0$

- For general purposes (any property that might change quickly, not just momentum), we define the Dirac Delta "function" as follows:

$$d_\tau(t) = (u_{-\tau}(t) - u_\tau(t))\frac{1}{2\tau}$$

$$\delta(t) = \lim_{\tau \to 0} d_\tau(t) = \begin{cases} \text{“}\infty\text{”} & \text{for } t = 0, \\ 0 & \text{for } t \neq 0. \end{cases}$$

$$g(t) = I_0 d_\tau(t)$$

- I_0 can be replaced by any type of quantity
- e.g. m_0 mass added to tank suddenly
- units of δ(t): 1 / time

Some facts about the Delta "function"

$$\int_a^b \delta(t)\ dt = 1 \qquad a < 0,\ b > 0$$ and = 0 otherwise.

$$\int_a^b f(t)\delta(t)\ dt = \lim_{\tau \to 0} \frac{1}{2\tau} \int_{-\tau}^{\tau} f(t)\ dt$$

$$= \lim_{\tau \to 0} \frac{F(\tau) - F(-\tau)}{2\tau} \qquad F'(t) = f(t)$$

$$= F'(0)\ = f(0)$$

$$\int_a^b f(t)\delta(t)\ dt = f(0) \qquad a < 0,\ b > 0$$ and = 0 otherwise.

$\delta(t - c) =$ shift of $\delta(t)$ by c

$$\int_a^b f(t)\delta(t - c)\ dt\ = \int_{a+c}^{b+c} f(u + c)\delta(u)\ du\ = f(c)$$ provided a<c<b.

Some facts about the Delta "function"

$$\int_a^b f(t)\delta(t-c)\ dt\ = f(c)$$

Laplace transform of delta function:

$$\begin{aligned}\mathcal{L}\{\delta(t-c)\} &= \int_0^\infty e^{-st}\delta(t-c)\ dt \\ &= \int_{-c}^\infty e^{-s(u+c)}\delta(u)\ du\ \ = e^{-sc} \text{ for } c > 0\end{aligned}$$

Relationship of delta function to other functions:

$$\frac{d}{dt}|t-c| =\ 2u_c(t) - 1$$

$$\frac{d}{dt}u_c(t) =\ \ \delta(t-c)$$