Today

- Transfer functions and convolution.
- Method of Undetermined Coefficients for any periodic function.
- Fourier Series and method of undetermined coefficients

Convolution

- We often end up with transforms to invert that are the product of two known transforms. For example,

$$Y(s) = \frac{2}{s^2(s^2+4)} = \frac{1}{s^2} \cdot \frac{2}{s^2+4}$$

- Can we express the inverse of a product in terms of the known pieces?

$$F(s)G(s) = \mathcal{L}\{??\}$$

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt \quad \longrightarrow \quad F(s) = \int_0^\infty e^{-s\tau} f(\tau)\ d\tau$$

$$G(s) = \int_0^\infty e^{-st} g(t)\ dt \quad \longrightarrow \quad G(s) = \int_0^\infty e^{-sw} g(w)\ dw$$

Convolution

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau)\ d\tau \int_0^\infty e^{-sw} g(w)\ dw$$

$$= \int_0^\infty e^{-sw} g(w) \int_0^\infty e^{-s\tau} f(\tau)\ d\tau\ dw$$

$$= \int_0^\infty g(w) \int_0^\infty e^{-s(\tau+w)} f(\tau)\ d\tau\ dw$$

Replace τ using $u = \tau + w$ where w is constant in the inner integral.

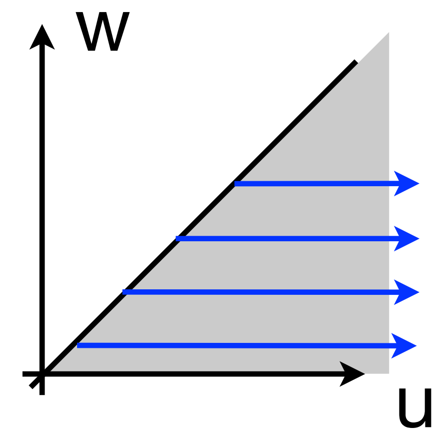
$$= \int_0^\infty g(w) \int_w^\infty e^{-s(u)} f(u-w)\ du\ dw$$

$$= \int_0^\infty \int_w^\infty e^{-su} g(w) f(u-w)\ du\ dw$$

$$= \int_a^b \int_c^d e^{-su} g(w) f(u-w)\ dw\ du$$

Convolution

- What are the correct values for a, b, c and d?



$$\int_0^{\infty}\int_w^{\infty} e^{-su} g(w) f(u-w)\ du\ dw$$

u= ; u= ; w=constant

$$= \int_a^b \int_c^d e^{-su} g(w) f(u-w)\ dw\ du$$

(A) Integrate in u from a=0 to b=∞ and in w from c=u, d=∞.

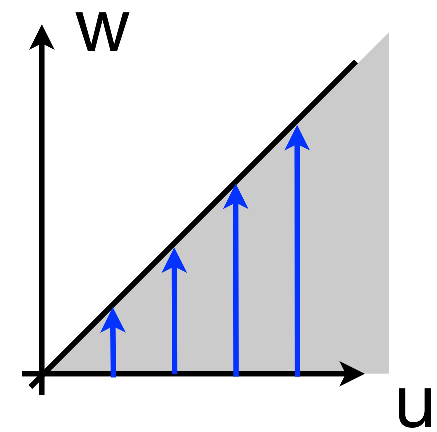
(B) Integrate in u from a=0 to b=w and in w from c=0 to d=∞.

(C) Integrate in u from a=0 to b=∞ and in w from c=0 to d=u.

(D) Integrate in u from a=0 to b=∞ and in w from c=w to d=∞.

(E) Huh?

Convolution



- What are the correct values for a, b, c and d?

$$\int_0^\infty \int_w^\infty e^{-su} g(w) f(u-w)\ du\ dw$$

$$= \int_a^b \int_c^d e^{-su} g(w) f(u-w)\ dw\ du$$

(A) Integrate in u from a=0 to b=∞ and in w from c=u, d=∞.

(B) Integrate in u from a=0 to b=w and in w from c=0 to d=∞.

★ (C) Integrate in u from a=0 to b=∞ and in w from c=0 to d=u.

(D) Integrate in u from a=0 to b=∞ and in w from c=w to d=∞.

(E) Huh?

Convolution

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau)\ d\tau \int_0^\infty e^{-sw} g(w)\ dw$$

$$= \int_0^\infty \int_0^u e^{-su} g(w) f(u-w)\ dw\ du$$

$$= \int_0^\infty e^{-su} \int_0^u g(w) f(u-w)\ dw\ du$$

$$= \int_0^\infty e^{-su} h(u)\ du\ = H(s)$$

The transform of a convolution is the product of the transforms.

$$h(t) = f * g(t) = \int_0^u g(w) f(t-w)\ dw$$

$$\Rightarrow H(s) = F(s)G(s)$$

where $h(u) = \int_0^u g(w) f(u-w)\ dw$

This is called the convolution of f and g. Denoted $f * g$.

Convolution

- To invert $Y(s) = \frac{1}{s^2} \cdot \frac{2}{s^2+4}$, we can use the fact that the inverse is the convolution of the inverses of the two pieces (instead of PFD...).

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin(2t)$$

$$f * g(t) = \int_0^u g(w)f(t-w)\ dw$$

$$f * g = g * f$$

$$\int_0^t f(t-w)g(w)\ dw = \int_0^t f(t)g(t-w)\ dw$$

$$y(t) =$$

☆ (A) $\int_0^t (t-w)\sin(2w)\ dw$

(B) $\int_0^t (t-w)\sin(2t)\ dw$

☆ (C) $\int_0^t w\sin(2(t-w))\ dw$

(D) $\int_0^t w\sin(2(w-t))\ dw$

Convolution

- Transfer functions

$$ay'' + by' + cy = g(t), \quad y(0) = 0,\ y'(0) = 0$$

$$Y(s) = \frac{1}{as^2 + bs + c} G(s)$$

- Define the transfer function for the ODE:

$$H(s) = \frac{1}{as^2 + bs + c}$$

Independent of g(t)!

$$y(t) = (h * g)(t)$$

- h(t) is called the impulse response because it solves (1) when g(t)=δ(t).

$$g(t) = \delta(t)$$

$$G(s) = e^{-0s} = 1$$

$$Y(s) = \frac{1}{as^2 + bs + c}$$

$$y_{IR}(t) = h(t) = \mathcal{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\}$$

Convolution

- Interpreting the transfer function in a model of memory.
- Your contact list got deleted. You are forced to memorize phone numbers. Let n(t) be the number of phone numbers you remember at time t. You forget numbers at a rate k. Finally, g(t) is the number of phone numbers per unit time that you memorize at time t.
- Equation: $n' = -kn + g(t)$
- Transform of n(t): $N(s) = \dfrac{G(s)}{s+k}$
- Transfer function: $H(s) = \dfrac{1}{s+k}$
- Impulse response: $h(t) = e^{-kt}$

$$n(t) = \int_0^t h(t-w)g(w)\ dw\ = \int_0^t e^{-k(t-w)}g(w)\ dw$$

Convolution

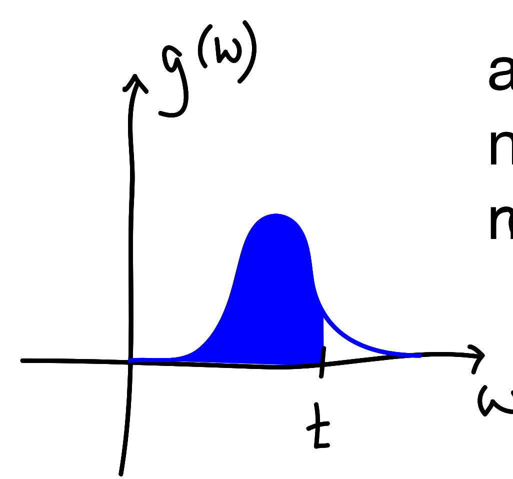
- Interpreting the transfer function in a model of memory.
- Your contact list got deleted. You are forced to memorize phone numbers. Let n(t) be the number of phone numbers you remember at time t. You forget numbers at a rate k. Finally, g(t) is the number of phone numbers per unit time that you memorize at time t.
- Equation: $$n' = -kn + g(t)$$
- If you memorize one phone number at t=0 (g(t)=δ(t)), h(t) tells you what's left of that memory at time t.

$$h(t) = e^{-kt}$$

- If you memorize numbers over time (some complicated g(t)),

$$n(t) = \int_0^t h(t-w)g(w)\ dw$$

$$= \int_0^t e^{-(t-w)}g(w)\ dw$$



all phone numbers ever memorized / still remembered

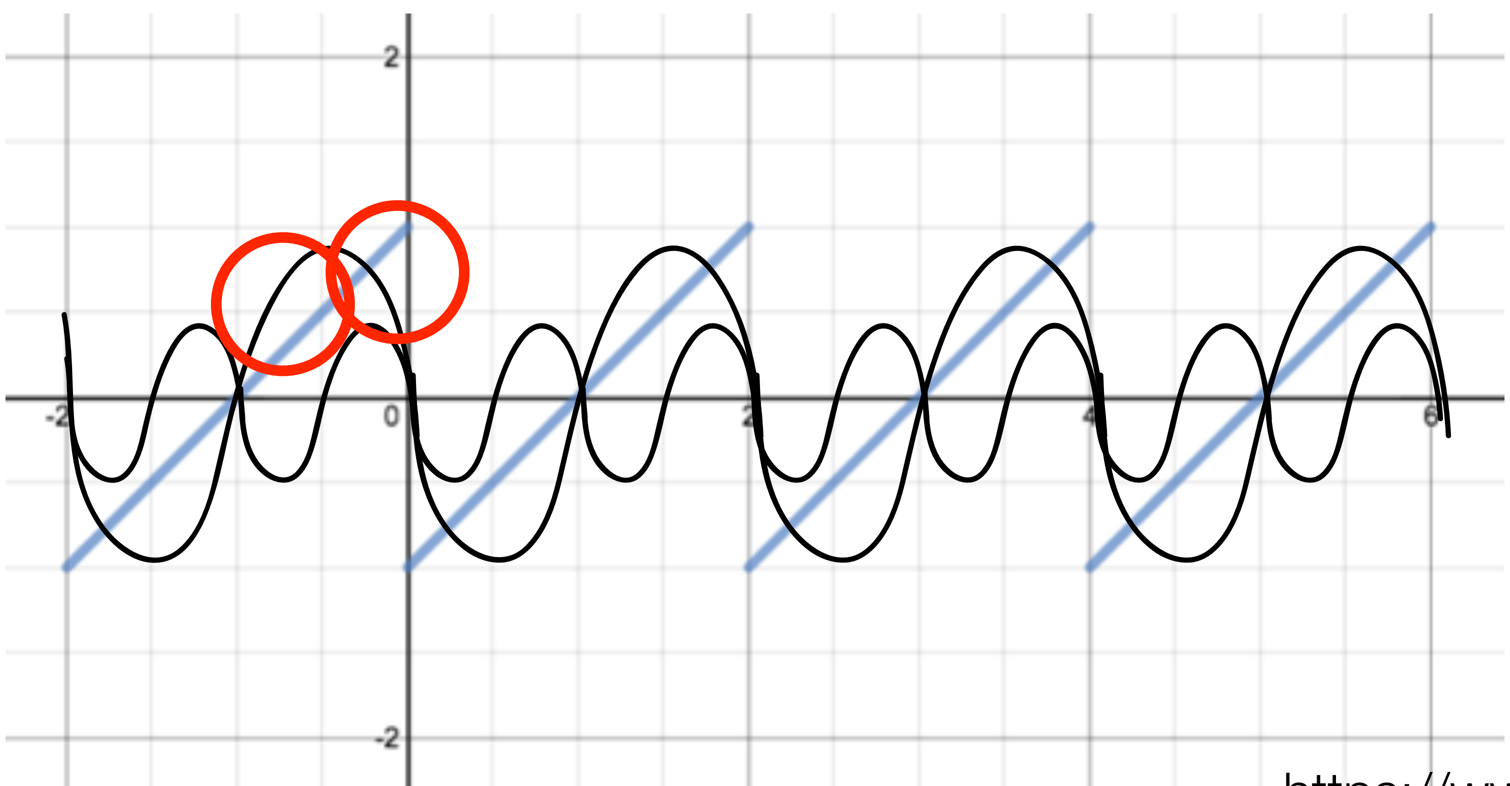
Fourier series

- Recall Method of Undetermined Coefficients for equations of the form

$$ay'' + by' + cy = f(t)$$

- Applicable for functions f(t) that are polynomials, exponentials, sin, cos and products of those.

- How about functions like this (periodic but not trig)?



- What if we could construct such functions using only sine and cosine functions?

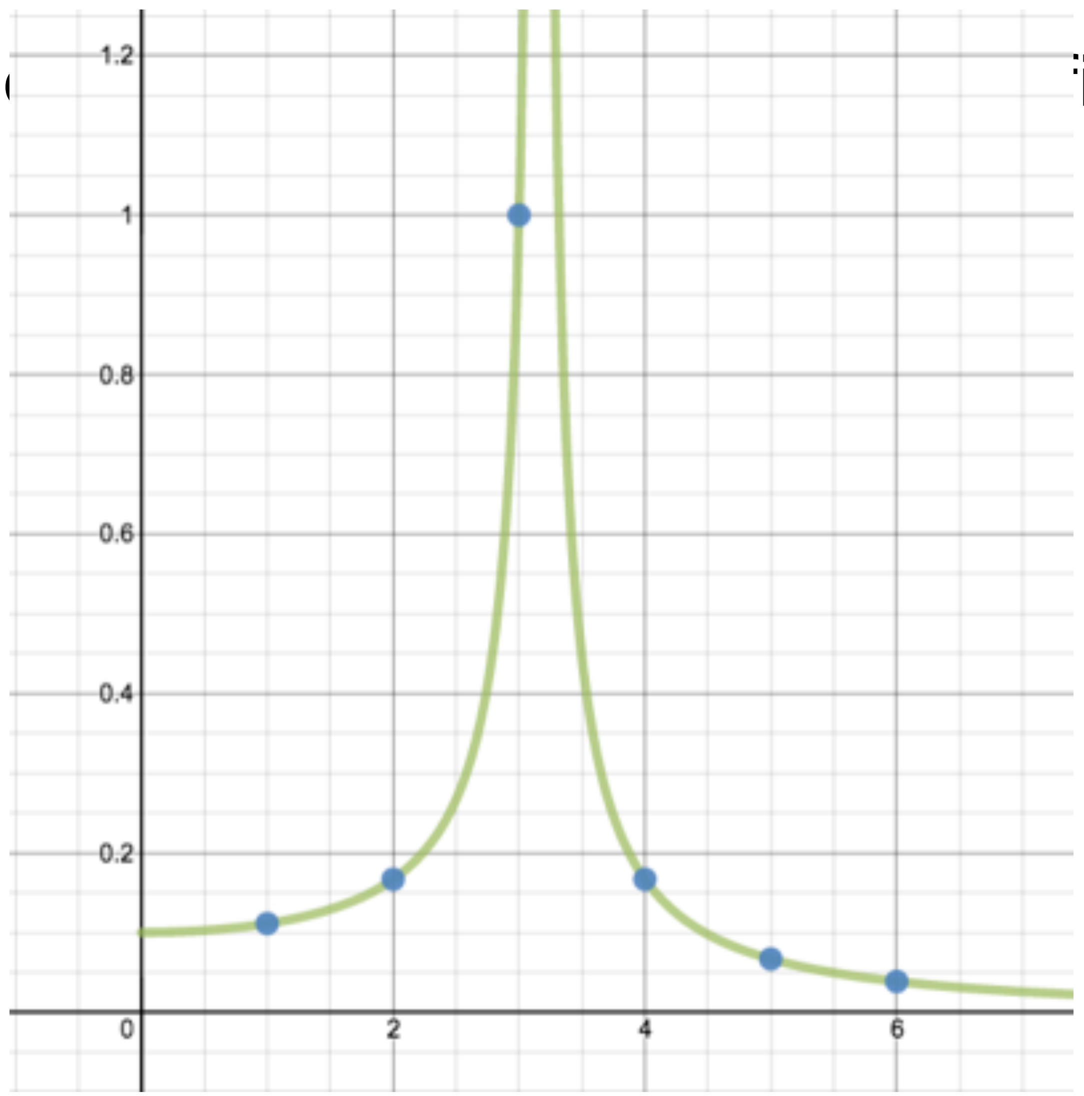
https://www.desmos.com/calculator/vecflcysms

Fourier series

- For the equation

$$y'' + 10y = \cos(t) + \cos(2t) + \cos(3t) + \cos(4t)$$

- what will be the … icient) in the solution?



(A) w = 1

(B) w = 2

(C) w = 3

(D) w = 4

(E) Don’t know. Explain please.

$$A(\omega) = \frac{1}{|\omega_0^2 - \omega^2|}$$

Fourier series

- Even if the coefficients decrease, for example,

$$y'' + 10y = \cos(t) + \frac{1}{2}\cos(2t) + \frac{1}{3}\cos(3t) + \frac{1}{4}\cos(4t) + \cdots$$

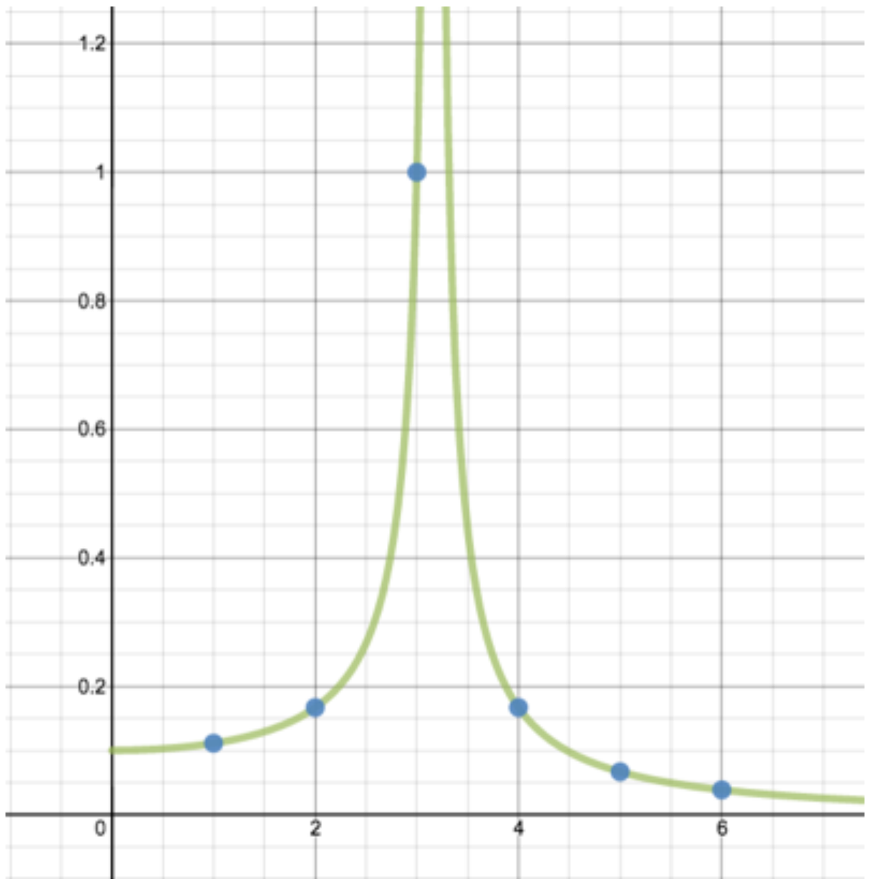
- a term with frequency close to resonance can still dominate the others:

(A) w = 1

(B) w = 2

★ (C) w = 3

(D) w = 4



$$A(\omega) = \frac{1}{|\omega_0^2 - \omega^2|}$$

(E) Don’t know. Explain please.

Fourier series (Method Undetermined Coefficients)

- Replace f(t) by a sum of trig functions, if possible:

$$ay'' + by' + cy = f(t) \stackrel{?}{=} A_0 + \sum_{n=1}^{N} a_n \cos(\omega_n t) + \sum_{n=1}^{N} b_n \sin(\omega_n t)$$

- For any f(t), how do we find the best choice of A_0, a_n, b_n?

https://www.desmos.com/calculator/vecflcysms

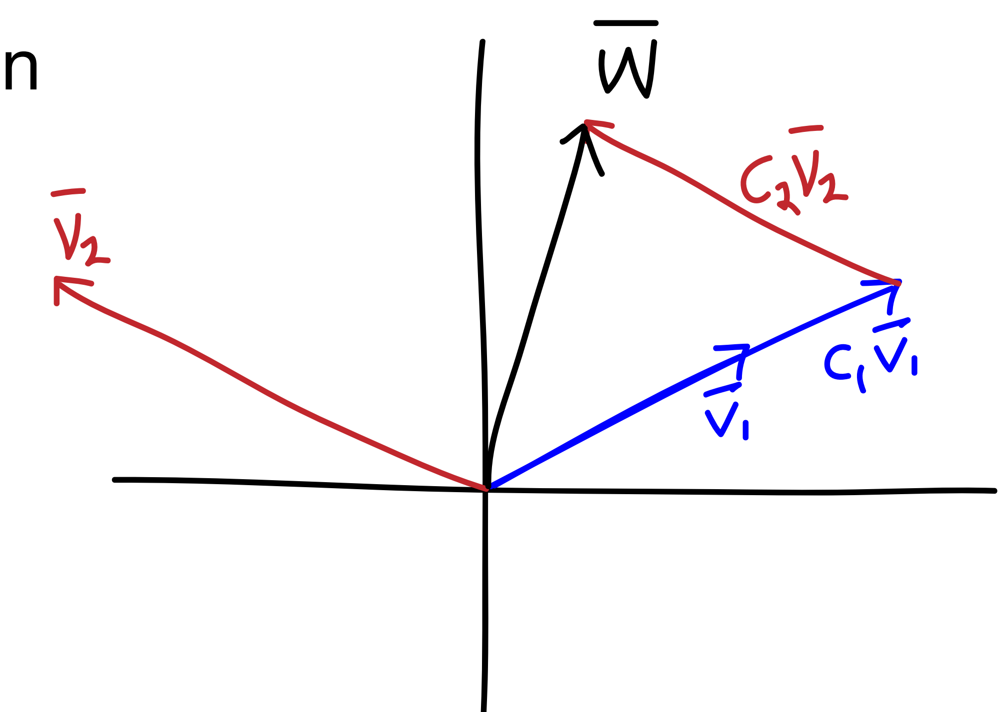
- This problem is closely related to an analogous vector problem: how do you choose c_1, c_2 so that $w = c_1 v_1 + c_2 v_2$?

- If v_1 and v_2 are perpendicular ($v_1 \circ v_2 = 0$), then

$$\mathbf{w} \circ \mathbf{v_1} = c_1 \mathbf{v_1} \circ \mathbf{v_1} + c_2 \mathbf{v_2} \circ \mathbf{v_1}$$

$$c_1 = \frac{\mathbf{w} \circ \mathbf{v_1}}{\mathbf{v_1} \circ \mathbf{v_1}}$$

$$\mathbf{v_1} \circ \mathbf{v_1} = ||\mathbf{v_1}||^2 \qquad c_2 = \frac{\mathbf{w} \circ \mathbf{v_2}}{\mathbf{v_2} \circ \mathbf{v_2}}$$



Fourier series (Method Undetermined Coefficients)

- For functions, define dot product as

$$g(t) \circ h(t) = \int_{\text{one period}} g(t)h(t)\ dt$$

- just like for vectors but indexed over all t instead of 1, 2, 3:

$$\mathbf{v} \circ \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Fourier series (Method Undetermined Coefficients)

- Back to our ODE, what do we choose for the w_n if f(t) has period T? Keep in mind that we want all the functions involved to have period T.

$$ay'' + by' + cy = f(t) \stackrel{?}{=} A_0 + \sum_{n=1}^{N} a_n \cos(\omega_n t) + \sum_{n=1}^{N} b_n \sin(\omega_n t)$$

(A) $w_n = \pi / T$

(B) $w_n = 2\pi / T$

(C) $w_n = n\pi / T$

☆ (D) $w_n = 2\pi n / T$

(E) Don't know. Explain please.

Once we find the coefficients, this will be the N-term Fourier polynomial representation of f(t). If we let N—> ∞ we get the Fourier series.

Draw graphs on doc cam.