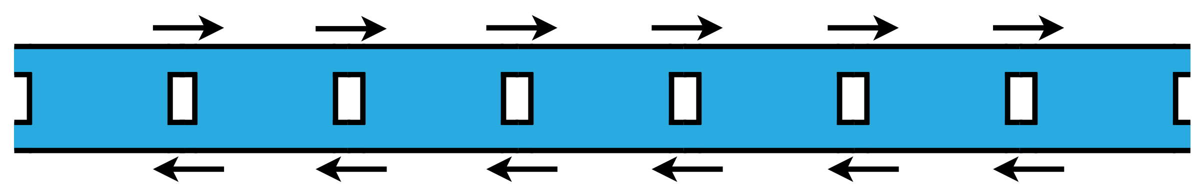
Today

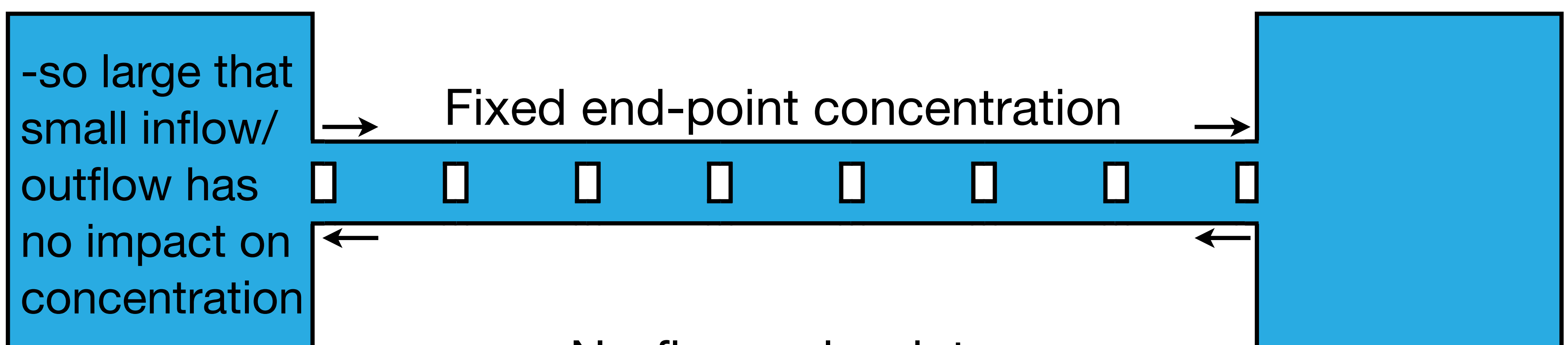
- Midterm - avg 72%, 11 fails, 12 in the 90%s, range 7%-100%.
- Chemical diffusion in a long narrow tube/rod.
  - Eigenvalues and eigenvectors in a discrete version (matrix problem).
  - Eigenvalues and eigenvectors in a continuous version (DE problem).
- Does the continuous version have a complete set of eigenvectors?
  - Fourier sine and cosine series

Diffusion in a long thin tube

- Treat the tube as if it were many small tanks connected by pipes.



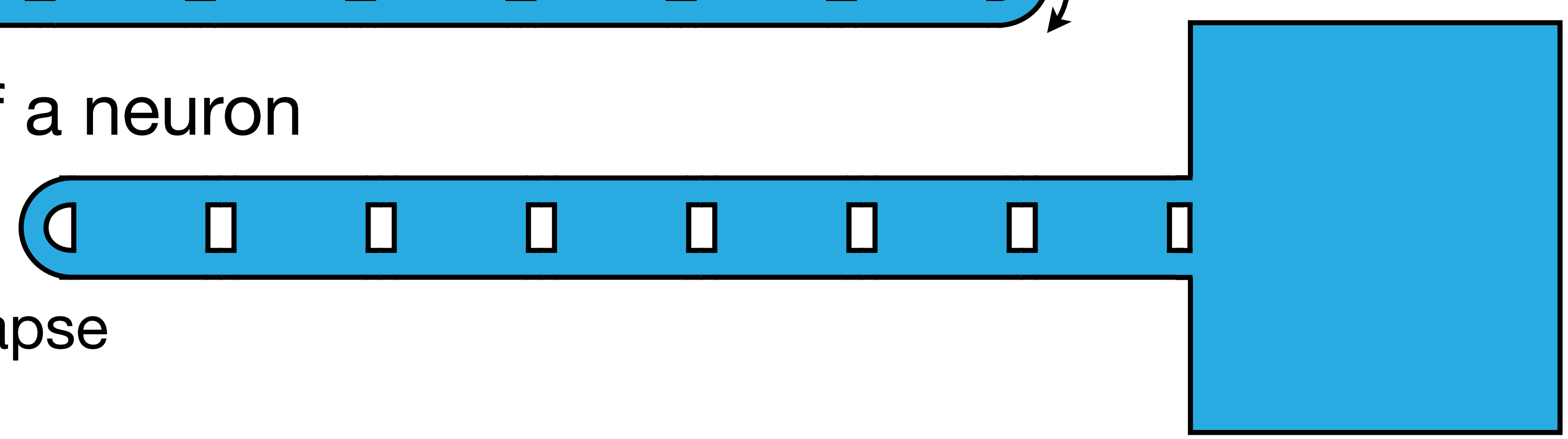
- Two common ways to deal with the ends of the tube:



No-flux end-points

-whatever leaves comes right back

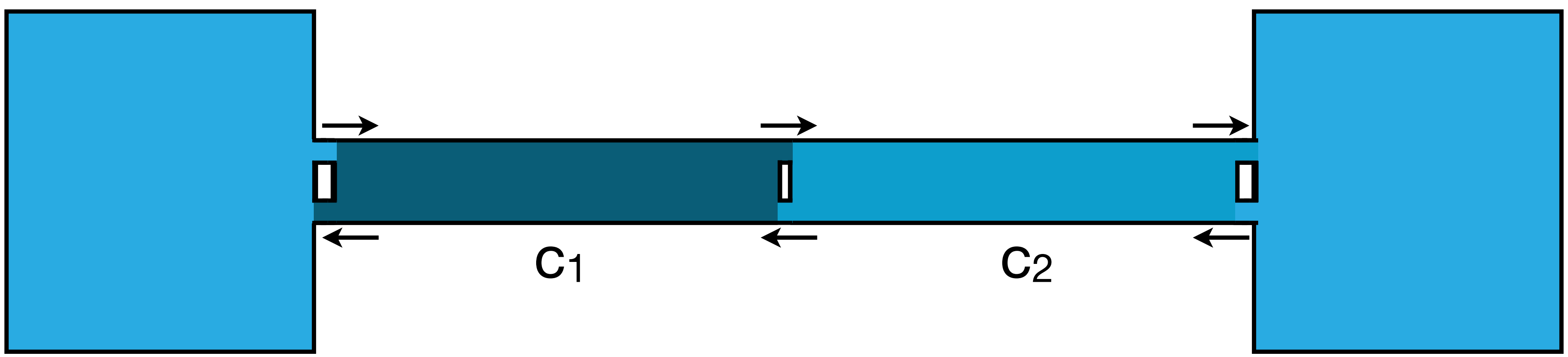
- Example: the axon of a neuron



Diffusion in a long thin tube

- What happens when a drop of dye is added to the tube:

- Two-tank approximation, fixed end-point concentration:



$$V\frac{dc_1}{dt} = kc_0 - kc_1 - kc_1 + kc_2$$

$$\frac{dc_1}{dt} = \frac{k}{V}(-2c_1 + c_2)$$

$$\frac{dc_1}{dt} = K(-2c_1 + c_2)$$

$$\frac{dc_2}{dt} = K(c_1 - 2c_2)$$

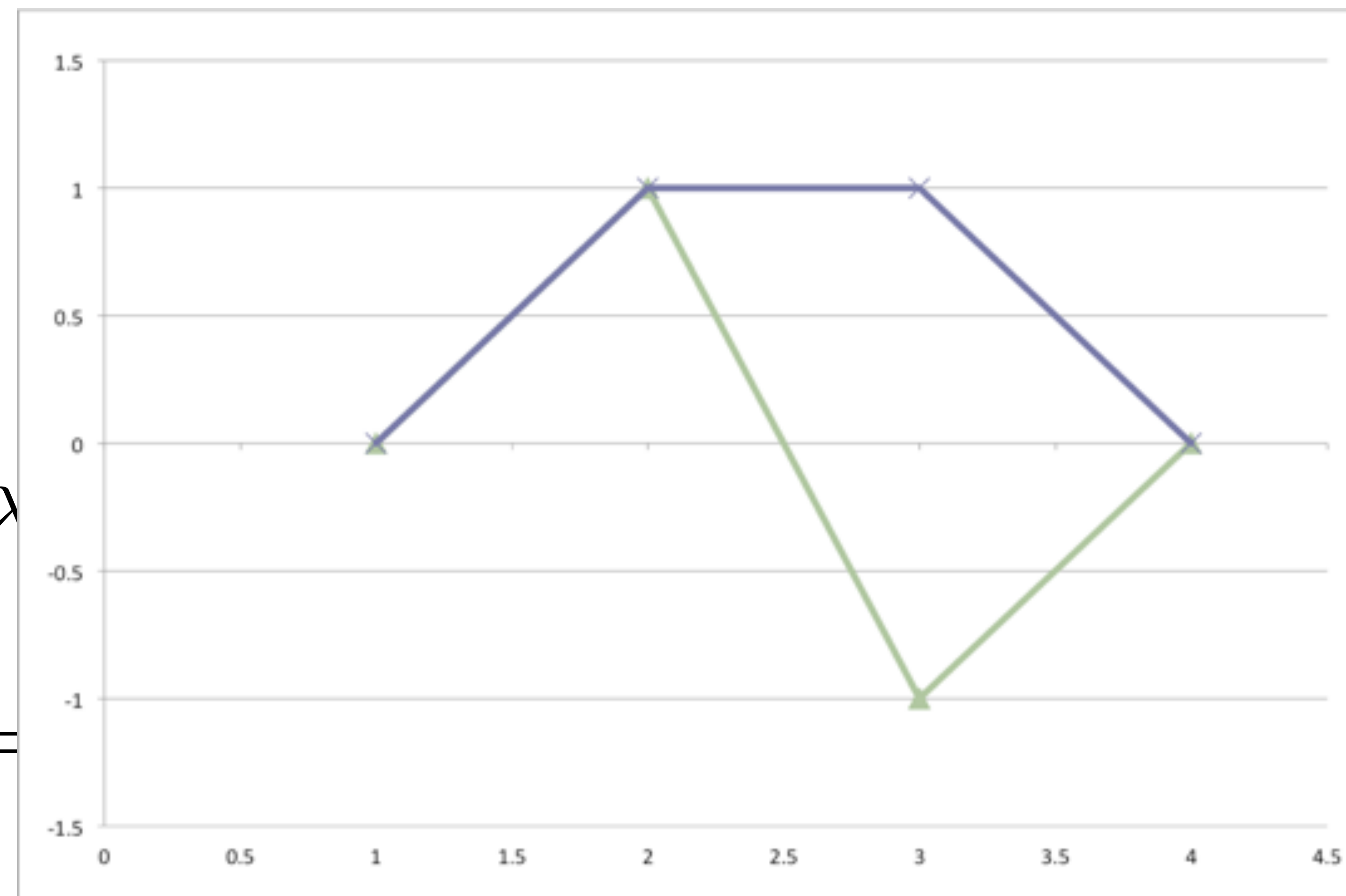
$$\frac{d\mathbf{c}}{dt} = K\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{c}$$

Diffusion in a long thin tube

$$\frac{d\mathbf{c}}{dt} = K\begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix}\mathbf{c}$$

$$\lambda_1 = -K \qquad \mathbf{v_1} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\lambda_2 = -3K \qquad \mathbf{v_2} = \begin{pmatrix}1\\-1\end{pmatrix}$$



$$\mathbf{c(t)} = Ae^{\lambda}$$

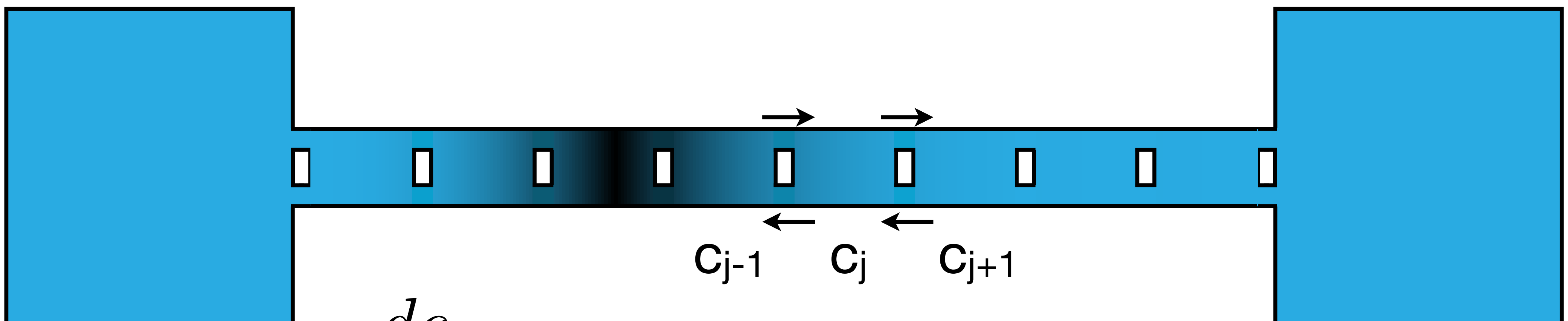
$$\mathbf{c(0)} =$$

- Average of two tanks (A) decays slowly at rate λ_1=-1 while difference decays quickly at rate λ_2=-3.

Diffusion in a long thin tube

- What happens when a drop of dye is added to the tube:

- Eight-tank approximation, fixed end-point concentration:

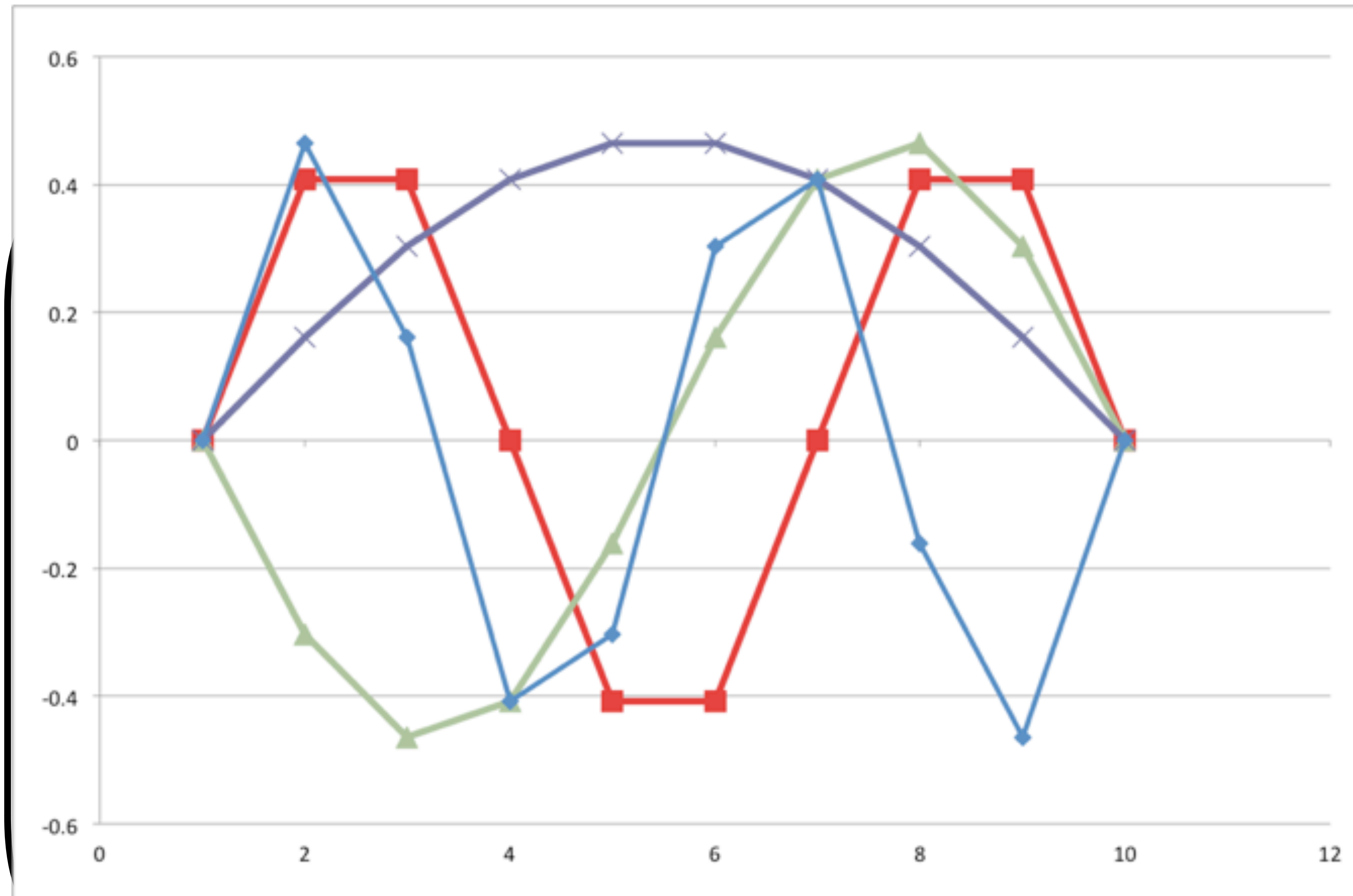


$$\frac{dc_j}{dt} = K(c_{j-1} - 2c_j + c_{j+1})$$

$$\frac{d\mathbf{c}}{dt} = K \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \mathbf{c}$$

Diffusion in a long thin tube

$$\lambda_1 = -64$$



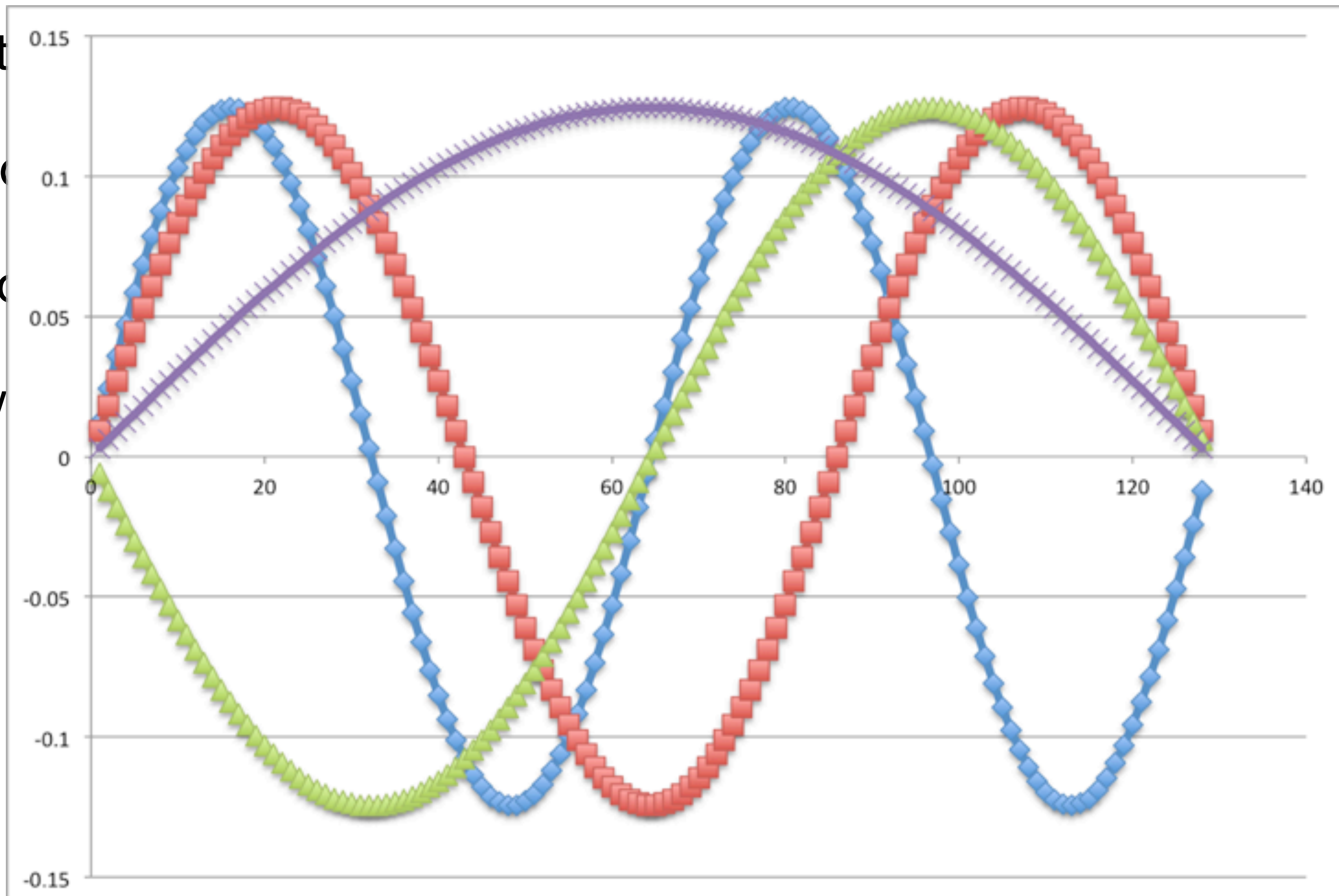
$$\mathbf{v_1} = \begin{pmatrix} 0 \\ 0.41 \\ 0.41 \\ 0 \\ -0.41 \\ -0.41 \\ 0 \\ 0.41 \\ 0.41 \\ 0 \end{pmatrix}$$

- Add these up to satisfy initial conditions. Each component decays at a different rate.

$$\mathbf{c(t)} = c_1 e^{\lambda_1 t}\mathbf{v_1} + c_2 e^{\lambda_2 t}\mathbf{v_2} + c_3 e^{\lambda_3 t}\mathbf{v_3} + c_4 e^{\lambda_4 t}\mathbf{v_4} + c_5 e^{\lambda_5 t}\mathbf{v_5} + c_6 e^{\lambda_6 t}\mathbf{v_6} + c_7 e^{\lambda_7 t}\mathbf{v_7} + c_8 e^{\lambda_8 t}\mathbf{v_8}$$

Diffusion in a long thin tube

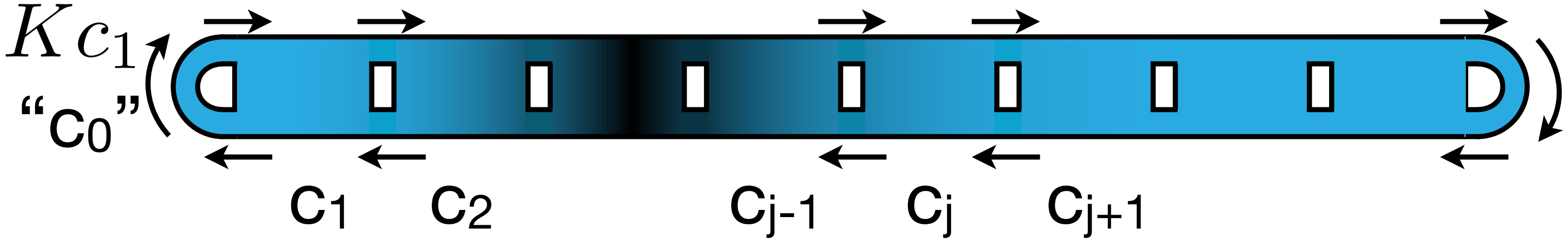
- 128-tank approximation, fixed end-point concentration:
- First
- Sec
- Thir
- Low … cay quickly.



Diffusion in a long thin tube

- Eight-tank approximation, no-flux end-points:

$Kc_0 = Kc_1$



$$\frac{dc_j}{dt} = K(c_{j-1} - 2c_j + c_{j+1})$$

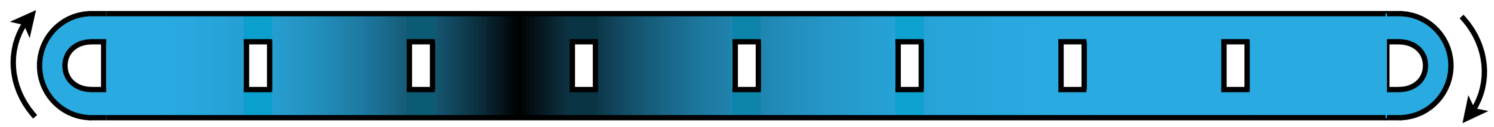
$$\frac{dc_1}{dt} = K(c_0 - 2c_1 + c_2)$$

$$\frac{dc_1}{dt} = K(-c_1 + c_2)$$

$$\frac{dc_8}{dt} = K(c_7 - c_8)$$

Diffusion in a long thin tube

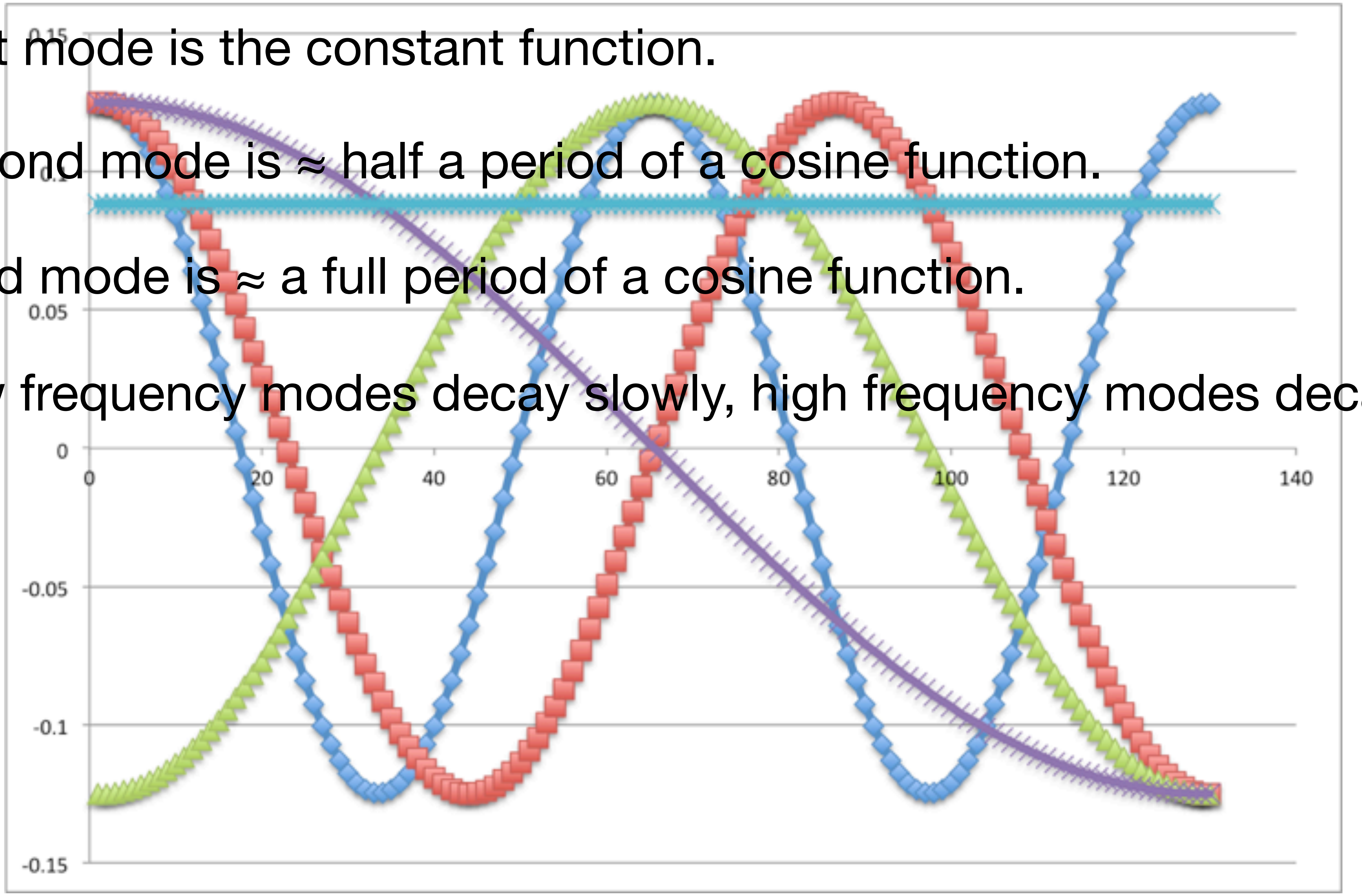
- Eight-tank approximation, no-flux end-points:



$$\frac{d\mathbf{c}}{dt} = K \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \mathbf{c}$$

Diffusion in a long thin tube

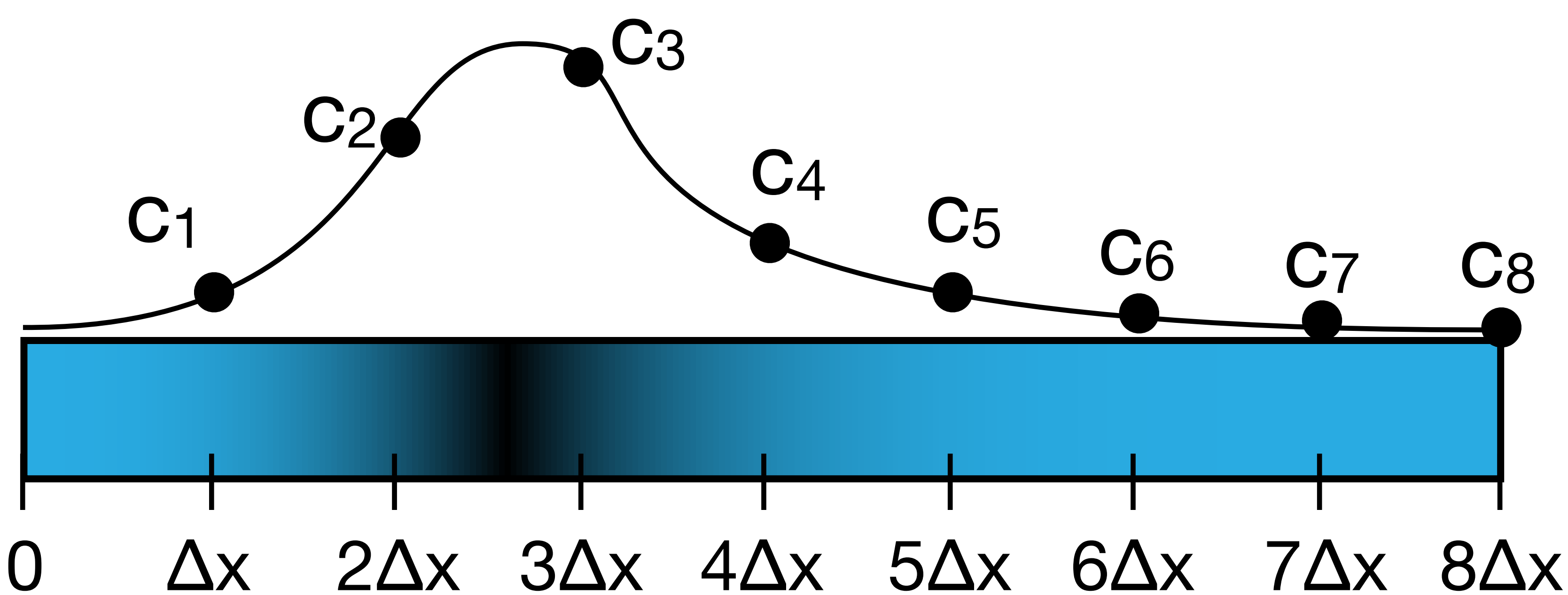
- 128-tank approximation, no-flux end-point:
- First mode is the constant function.
- Second mode is ≈ half a period of a cosine function.
- Third mode is ≈ a full period of a cosine function.
- Low frequency modes decay slowly, high frequency modes decay quickly.



Diffusion in a long thin tube

- In the limit of an infinite number of tiny tanks...

- Suppose there is a function $c(x,t)$ such that $c_j(t) = c(j\Delta x,t)$.



$$\frac{dc_j}{dt} = K(c_{j-1} - 2c_j + c_{j+1})$$

- Want to replace this equation by one for the function $c(x,t)$...

Diffusion in a long thin tube

$$\frac{dc_j}{dt} = K(c_{j-1} - 2c_j + c_{j+1})$$

- Recall Taylor series:

$$f(x + \Delta x) \approx \dots \frac{1}{2} f''(x)\Delta x^2$$

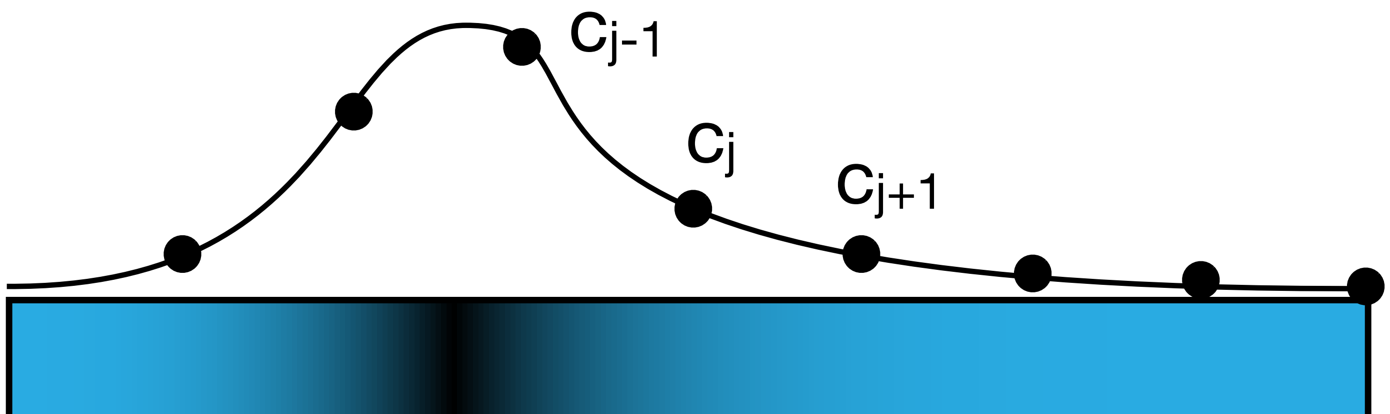
$$c_{j+1}(t) = c(x + \Delta \dots \qquad (x = j\Delta x)$$

$$\approx c(x,t) + \dots \frac{\dots}{dx} \dots \frac{\dots}{2}\frac{\dots}{dx^2} c(x,t)\Delta x^2$$

The Diffusion Equation

$$\frac{dc}{dt} = D\frac{d^2 c}{dx^2}$$

$$c_{j-1}(t) \approx c(x,t) - \frac{d}{dx}c(x,t)\Delta x + \frac{1}{2}\frac{d^2}{dx^2}c(x,t)\Delta x^2$$

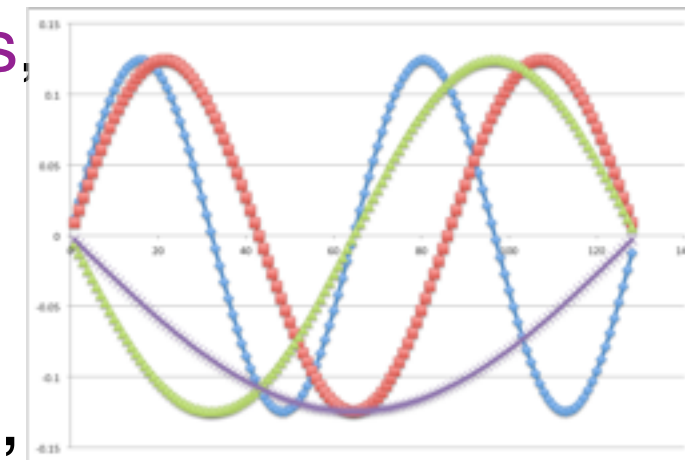


$$-2c_j(t) = -2c(x,t)$$

$$\frac{dc}{dt} = K\Delta x^2 \frac{d^2 c}{dx^2}$$

Diffusion in a long thin tube

- What happens to the end-point conditions? Called boundary conditions.
- For fixed end-point concentrations, called Dirichlet BCs,

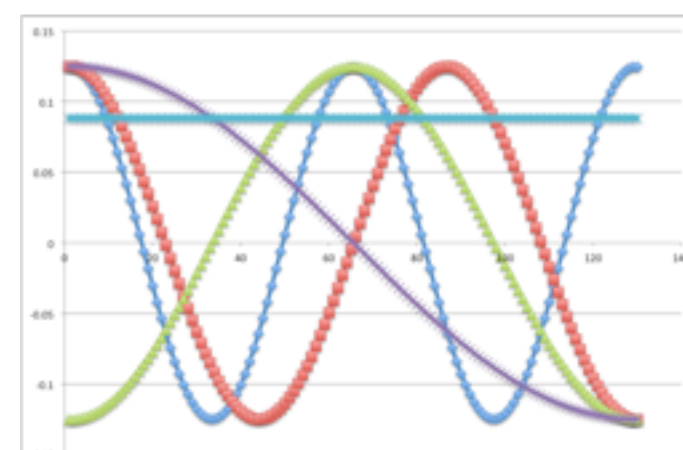


$$c(0,t) = c(L,t) = c_0 = 0$$

- For no-flux end-points, called Neumann or no-flux BCs,

$$c_1 - c_0 = c(\Delta x, t) - c(0,t) = 0$$

$$\frac{c(\Delta x, t) - c(0,t)}{\Delta x} = 0$$



$$\frac{dc}{dx}(0,t) = 0 \qquad \frac{dc}{dx}(L,t) = 0$$

Diffusion in a long thin tube

- Eigenvalues and eigenvectors for a partial differential equation:
- For the matrix equation $\mathbf{c}' = A\mathbf{c}$, find all eigenvalues λ and eigenvectors **c**: $A\mathbf{c} = \lambda\mathbf{c}$.
- For the PDE

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

find all eigenvalues λ and eigen"vectors" c(x):

$$D\frac{d^2c}{dx^2} = \lambda c$$

$$\lambda = ??$$

If λ>0,

$$c(x) = e^{\sqrt{\frac{\lambda}{D}}x}$$

or

$$c(x) = e^{-\sqrt{\frac{\lambda}{D}}x}$$

If λ<0,

$$c(x) = \sin\left(\sqrt{\frac{-\lambda}{D}}x\right)$$

or

$$c(x) = \cos\left(\sqrt{\frac{-\lambda}{D}}x\right)$$

Diffusion in a long thin tube

- To find λ, impose appropriate boundary conditions.
- If the physical system has fixed end-point concentrations, use Dirichlet BCs and find all functions with corresponding λ that work.
- The exp functio … onditions so

$$\lambda = -\frac{P^2\pi^2 D}{L^2}$$ for all integers P≠0

$$c(x) = \mathrm{s} \ldots \left(\sqrt{\frac{-\lambda}{D}}x\right)$$

- Which of following satisfies $c(0) = 0$ and $c(L) = 0$?

$$\sin\left(\sqrt{\frac{-\lambda}{D}}x\right)$$

(A) with $\lambda = -\frac{2\pi D}{L}$

(B) with $\lambda = -\frac{8\pi^2 D}{L^2}$

☆(C) with $\lambda = -\frac{4\pi^2 D}{L^2}$

☆(D) with $\lambda = -\frac{16\pi^2 D}{L^2}$

Diffusion in a long thin tube

- How to solve an Initial Value Problem for the Diffusion Equation?

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

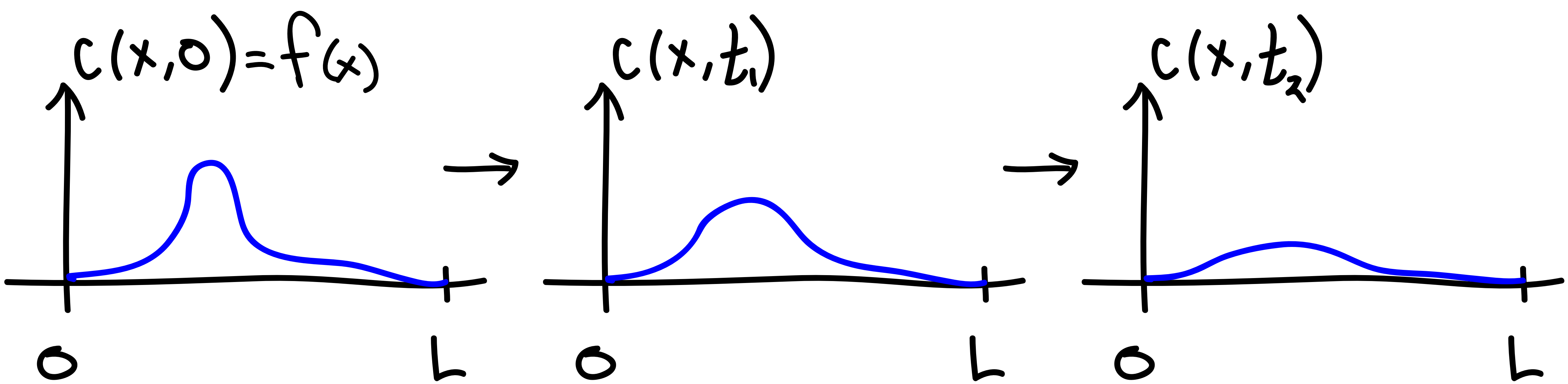
PDE

$$c(L,t) = 0$$
$$c(0,t) = 0$$

boundary conditions

$$c(x,0) = f(x)$$

initial condition



Diffusion in a long thin tube

- How to solve an Initial Value Problem for the Diffusion Equation?

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

PDE

$$c(L,t) = 0$$
$$c(0,t) = 0$$

boundary conditions

$$c(x,0) = f(x)$$

initial condition

$$c(x,t) = A_1 e^{\lambda_1 t}\sin(\omega_1 x) + A_2 e^{\lambda_2 t}\sin(\omega_2 x) + A_3 e^{\lambda_3 t}\sin(\omega_3 x) + \cdots$$

where A_p are unknown constants to be determined by the IC, and

$$\lambda_p = -\frac{p^2\pi^2 D}{L^2} \quad \text{and} \quad \omega_p = \frac{p\pi}{L}$$

Diffusion in a long thin tube

- To find λ, impose appropriate boundary conditions.
- If the physical system has no-flux end-points, use Neumann BCs and find all sin/cos functions with corresponding λ that work.
- The exp functions can't satisfy Dirichlet (or Neumann) conditions so

$$c(x) = \mathrm{s}$$

$$\lambda = -\frac{P^2\pi^2 D}{L^2}$$ for all integers P≠0

$$\sqrt{\frac{-\lambda}{D}x}\right)$$

- Which of following satisfies $\frac{dc}{dx}(0) = 0$ and $\frac{dc}{dx}(L) = 0$?

$$\cos\left(\sqrt{-\frac{\lambda}{D}}x\right)$$

☆(A) with $\lambda = 0$

(B) with $\lambda = -\left(P + \frac{1}{2}\right)\frac{\pi^2}{L^2}D$

☆(C) with $\lambda = -\frac{4\pi^2 D}{L^2}$

☆(D) with $\lambda = -\frac{16\pi^2 D}{L^2}$

Diffusion in a long thin tube

- How to solve an Initial Value Problem for the Diffusion Equation?

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

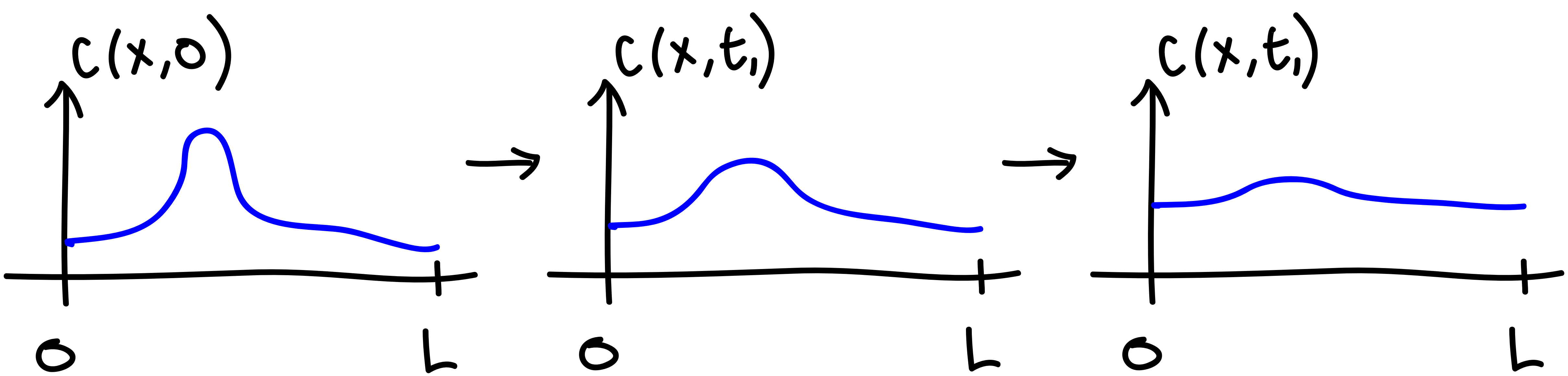
$$\frac{dc}{dx}(0,t) = 0$$
$$\frac{dc}{dx}(L,t) = 0$$

$$c(x,0) = f(x)$$

PDE

boundary conditions

initial condition



Diffusion in a long thin tube

- How to solve an Initial Value Problem for the Diffusion Equation?

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

$$\frac{dc}{dx}(0,t) = 0$$
$$\frac{dc}{dx}(L,t) = 0$$

$$c(x,0) = f(x)$$

PDE

boundary conditions

initial condition

$$c(x,t) = A_0 + A_1 e^{\lambda_1 t}\cos(\omega_1 x) + A_2 e^{\lambda_2 t}\cos(\omega_2 x) + A_3 e^{\lambda_3 t}\cos(\omega_3 x) + \cdots$$

where A_p are unknown constants to be determined by the IC, and

$$\lambda_p = -\frac{p^2\pi^2 D}{L^2} \quad \text{and} \quad \omega_p = \frac{p\pi}{L}$$