# MATH 256 tutorial – Midterm 2 review problems.

1. A undamped mass-spring system with mass 1 kg and spring constant 16 kg/s$^2$, is initially at rest. At $t = 3$, a linearly increasing force is applied until the force reaches $F_0 = 10$ N at $t = 8$. After that moment, the force remains constant at that level ($F_0$).

   (a) Write down the forcing function for this scenario using Heaviside functions in the form $u_c(t)g(t-c)$.

   (b) Write down the ODE for this mass-spring system subject to the given forcing function.

   (c) Solve the ODE.

2. Two tanks are connected by pipes. They initially contain large quantities of salt. Freshwater is added to the tanks so that the volumes of water are constant. The mass of salt in each tank is given by the system of equations

$$\frac{d}{dt}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

   where time is measured in minutes. How long does it take for the concentration in both tanks to decrease to less than one tenth of their original values? (You may use the approximation $e^{-1} \approx 1/3$ to do this without a calculator.)

   (a) Less than 1 minute

   (b) 1-2 minutes

   (c) 2-3 minutes

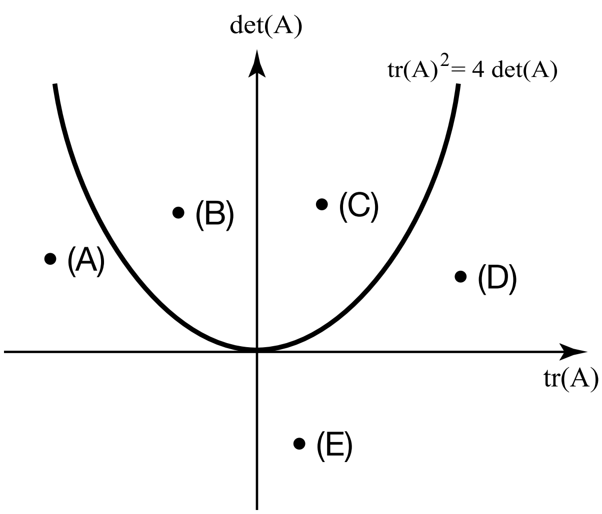
   (d) 3-5 minutes

3. Consider the system of equations given in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha+1 & 1 \\ 1 & \alpha-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

   (a) Using the letters from the diagram below, list (in order) the regions of the trace/determinant plane that the system moves through as $\alpha$ goes from $-\infty$ to $\infty$.

   (b) Find the value of alpha at each transition.




4. The motion of a forced tuning fork satisfies the equation $y'' + 2y' + 9y = 5\cos(\omega t)$. The Method of Undetermined Coefficients gives a particular solution of

$$y_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

   where

$$A = \frac{5(9-\omega^2)}{4\omega^2 + (9-\omega^2)^2} \quad \text{and} \quad B = \frac{10\omega}{4\omega^2 + (9-\omega^2)^2}.$$

At what frequency $\omega$ does the tuning fork vibrate with largest amplitude? Using a formula for the amplitude of $y_p$ is acceptable. Any claims (e.g. "$\omega = 42$ is a max.") must be justified.

**Hint:** Recall that taking the square root of a function does not change the location of its minima and the reciprocal of a function has maxima wherever the original function has minima.

5. An ectotherm is an animal that does not regulate its own temperature but instead depends on the environmental temperature to determine its body temperature. The rate of change of an ectotherm's body temperature satisfies Newton's Law of Cooling:

$$B' = k(E(t) - B)$$

where $B(t)$ is the animal's body temperature, $E(t)$ is the environmental temperature and $k$ is a positive constant that is determined by how well insulated the animal is - better insulation means a smaller $k$ value. The environmental temperature varies daily as $E(t) = 20 + 10\cos(t)$ in degrees Celsius.

   (a) Find the general solution $B(t)$. Hint: This can be done using an integrating factor or the Method of Undetermined Coefficients. The latter approach is the simpler one in this case.

   (b) What is the amplitude of the oscillatory part of the general solution? Your answer should depend on $k$.

   (c) Give an approximation (independent of $k$) for the phase difference between the oscillatory part of the animal's temperature and the environmental temperature when the animal is very poorly insulated ($k$ very large)?

   (d) Give an approximation (independent of $k$) for the phase difference between the oscillatory part of the animal's temperature and the environmental temperature when the animal is very well insulated ($k$ very small)?

6. Omar is a recovering heroine addict. As part of his recovery program, he uses a chemical patch on his arm that delivers methadone in an oscillatory manner to avoid the adaptive response associated with constant delivery. The equation for the concentration of methadone in Omar's blood stream is

$$M' = P(t) - kM$$

where $P(t) = \alpha(1 + \cos(\omega t))$ is the rate in mg/min at which the patch delivers methadone and $k$ is a positive constant that is determined by how quickly the drug is metabolised.

   (a) Find the general solution $M(t)$. Hint: This can be done using an integrating factor or the Method of Undetermined Coefficients. The latter approach is the simpler one in this case.

   (b) What is the amplitude of the oscillatory part of the general solution? Your answer should depend on $k, \alpha$ and $\omega$. Simplify your answer as much as possible.

   (c) Express the oscillatory part of the solution as a single cosine expression for the case of $\omega = k$?