Today

- Teaching evals (10 min)
- Diffusion equation examples and summary
- Please fill out poll on Facebook to influence office hour and review dates.

Nonhomogeneous boundary conditions

- Find the solution to the following problem:

$$u_t = 4u_{xx}$$
$$u(0,t) = 9$$
$$u(2,t) = 5$$
$$u(x,0) = \sin\frac{3\pi x}{2}$$

(A) $u(x,t) = e^{-9\pi^2 t}\sin\frac{3\pi x}{2}$

(B) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2}$

★ (C) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2} + 9 - 2x$

(D) $u(x,t) = e^{-9\pi^2 t}\sin\frac{3\pi x}{2} + 9 - 2x$

where $b_n = ?$

Nonhomogeneous boundary conditions

- Find the solution to the following problem:

$$u_t = 4u_{xx}$$
$$u(0,t) = 9$$
$$u(2,t) = 5$$
$$u(x,0) = \sin\frac{3\pi x}{2}$$

(A) $b_n = \int_0^2 \sin\frac{3\pi x}{2}\cos\frac{n\pi x}{2}\,dx$

(B) $b_n = \int_0^2 \sin\frac{3\pi x}{2}\sin\frac{n\pi x}{2}\,dx$

★ (C) $b_n = \int_0^2 \left(\sin\frac{3\pi x}{2} - 9 + 2x\right)\sin\frac{n\pi x}{2}\,dx$

(D) $b_n = \int_0^2 \left(\sin\frac{3\pi x}{2} + 9 - 2x\right)\sin\frac{n\pi x}{2}\,dx$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2} + 9 - 2x$$

Nonhomogeneous boundary conditions

- How would you solve this one?

$$u_t = 4u_{xx}$$

$$\left.\frac{du}{dx}\right|_{x=0,2} = -2$$

$$u(x,0) = \cos\frac{3\pi x}{2}$$

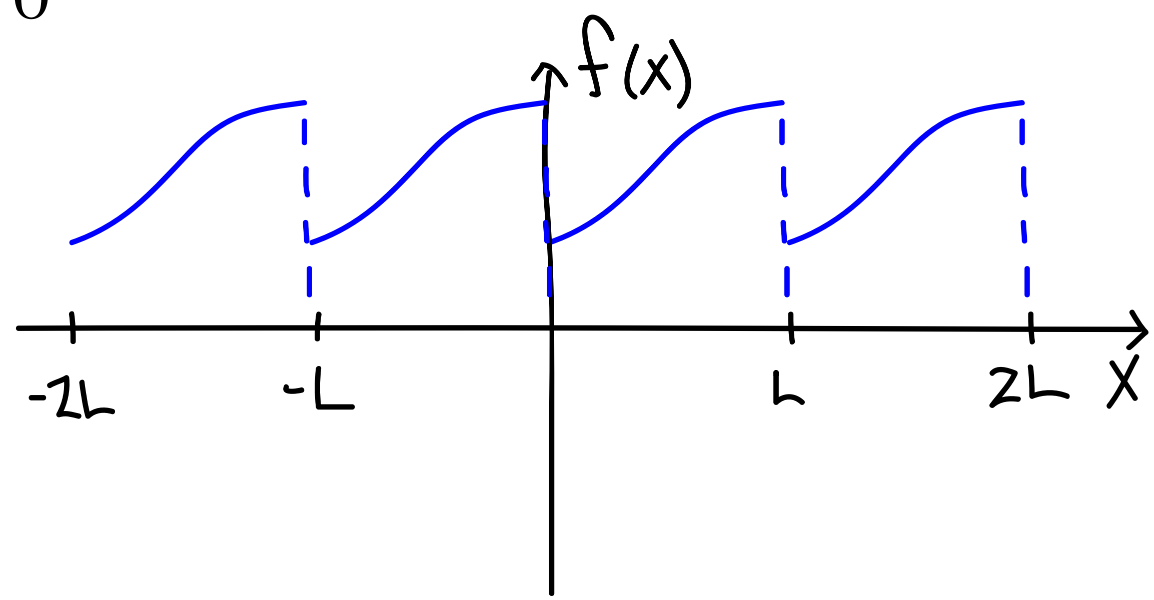
Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$

- Extend f(x) to all reals as a periodic function.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

- All coefficients will be non-zero. Not particularly useful for solving the BCs.

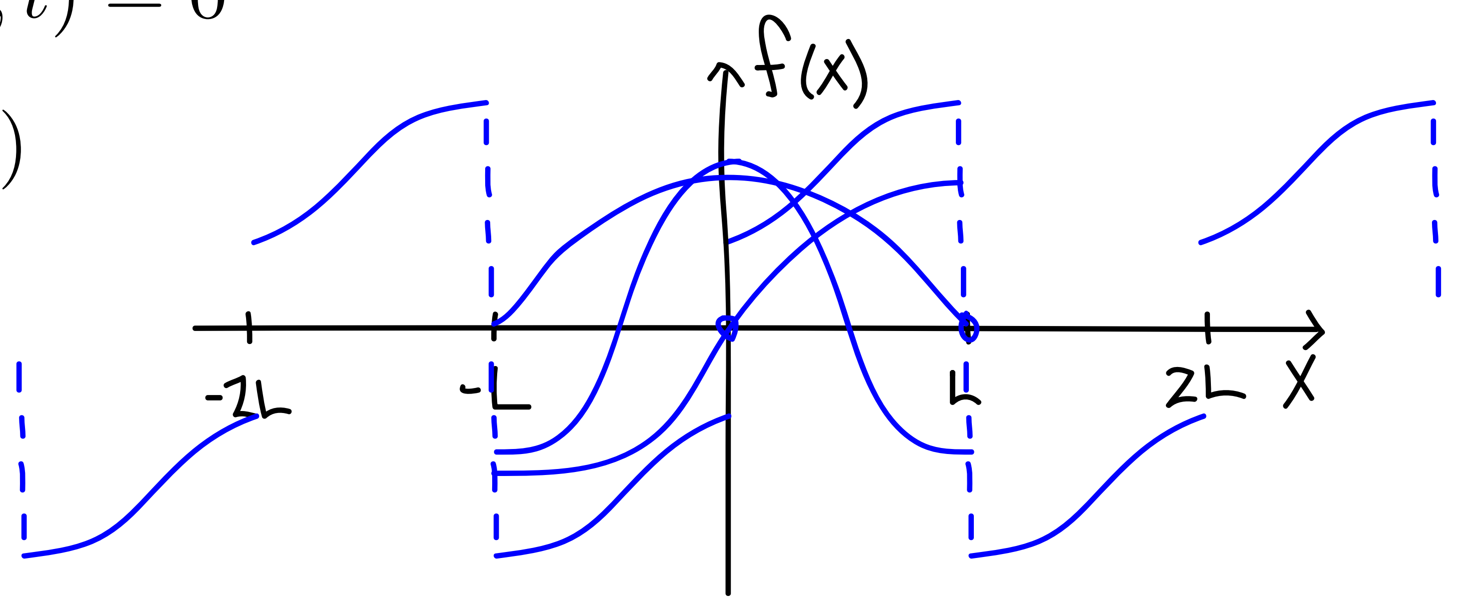
Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$

- Extend to -L as an odd function and then to all reals as a periodic function.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

- Cosine coefficients will be zero because f(x) is odd about x=0 and cosine is even. Useful for solving the Diffusion equation with Dirichlet BCs.

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{ext}(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f_{ext}(x)\sin\frac{n\pi x}{L}\,dx$$

$$a_n = 0 \qquad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Review of solutions to the Diffusion Equation

$u_t = Du_{xx}$

$u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 Dt/L^2} \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Review of solutions to the Diffusion Equation

$$u_t = D u_{xx}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,L} = 0$$

$$u(x,0) = f(x)$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 Dt/L^2} \cos\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx$$

Review of solutions to the Diffusion Equation

$$u_t = Du_{xx}$$
$$u(0,t) = a$$
$$u(L,t) = b$$
$$u(x,0) = f(x)$$

$$u(x,t) = a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/L^2} \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - a - \frac{b-a}{L}x \right) \sin\frac{n\pi x}{L}\, dx$$

- Adding the linear function to the usual solution to the Dirichlet problem ensures that the BCs are satisfied without changing the fact that it satisfies the PDE.

Review of solutions to the Diffusion Equation

$$u_t = D u_{xx}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,L} = a$$

$$u(x,0) = f(x)$$

$$u_{ss}(x) = ax + B$$

$$B = \frac{1}{L}\int_0^L f(x)\ dx - \frac{1}{2}aL$$

$$u(x,t) = ax + B + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 Dt/L^2} \cos\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L (f(x) - ax - B)\cos\frac{n\pi x}{L}\ dx$$