Today

- Complex number review and Euler’s formula
- Complex roots to the characteristic equations
- Repeated roots to the characteristic equations

Tutorial poll

(A) Post worksheet online on Friday, print and hand in during tutorial.

(B) Post worksheet online on Friday, get from TA and hand in during tutorial.

(C) Hand out worksheet during tutorial, hand in during Tuesday class.

(D) As currently done.

Complex number review

- We define a new number: $i = \sqrt{-1}$
- Before, we would get stuck solving any equation that required square-rooting a negative number. No longer.
- e.g. The solutions to $x^2 - 4x + 5 = 0$ are $x = 2 + i$ and $x = 2 - i$
- For any equation, $ax^2 + bx + c = 0$, when $b^2 - 4ac < 0$, the solutions have the form $x = \alpha \pm \beta i$ where α and β are both real numbers.
- For α+βi, we call α the real part and β the imaginary part.

Complex number review

- Adding two complex numbers:

$$(a+bi)+(c+di) = \underbrace{a+c}_{} + \underbrace{(b+d)}_{}i$$

- Multiplying two complex numbers:

$$(a+bi)(c+di) = \underbrace{ac-bd}_{} + \underbrace{(ad+bc)}_{}i$$

- Dividing by a complex number:

$$(a+bi)/(c+di) = (a+bi)\frac{1}{(c+di)}$$

  - What is the inverse of c+di?

Complex number review

- What is the inverse of c+di written in the usual complex form p+qi?

(A) $c - di$

(B) $\dfrac{c+di}{c^2+d^2}$

★ (C) $\dfrac{c-di}{c^2+d^2}$

(D) $\dfrac{1}{c-di}$

$$(c+di)\frac{c-di}{c^2+d^2} = \frac{c^2+d^2-(cd-dc)i}{c^2+d^2} = 1$$

- Dividing by a complex number:

$$(a+bi)/(c+di) = (a+bi)\frac{c-di}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2}$$

Complex number review

- Definitions:

  - Conjugate - the conjugate of $a + bi$ is

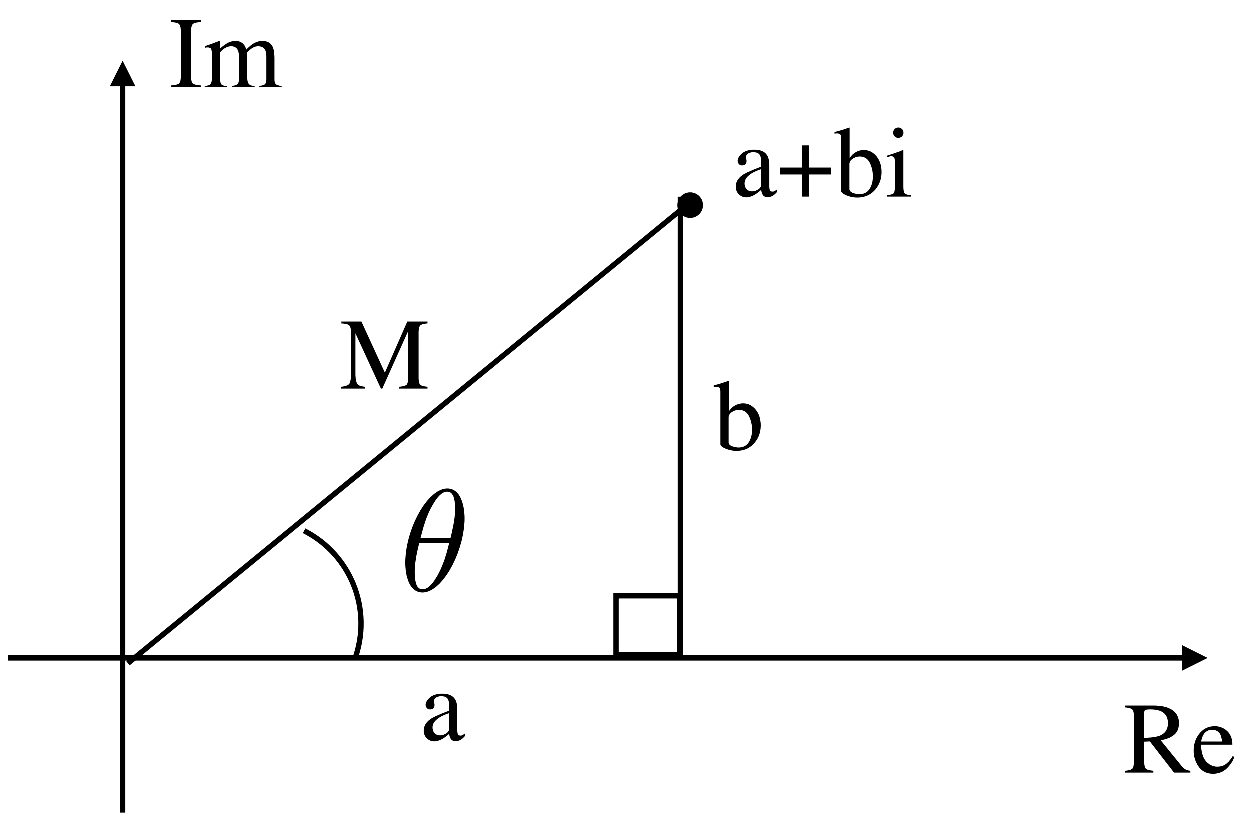
$$\overline{a + bi} = a - bi$$

  - Magnitude - the magnitude of $a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}$$

Complex number review

- Geometric interpretation of complex numbers
    - e.g. $a + bi$



$$a = M \cos \theta$$

$$b = M \sin \theta$$

$$M = \sqrt{a^2 + b^2}$$

$$\theta = \arctan \left(\frac{b}{a} \right)$$

$$a + bi = M(\cos \theta + i \sin \theta)$$

θ is sometimes called the argument or phase of $a + bi$.

Complex number review

- Toward Euler's formula
    - Taylor series - recall that a function can be represented as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

- What function has Taylor series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

★ (A) cos x ★ (C) e^x

★ (B) sin x (D) ln x

Complex number review

- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

$$\cos\theta + i\sin\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$\boxed{-1 = i^2}$$

$$= 1 + i\theta + (-1)\frac{\theta^2}{2!} + (-1)i\frac{\theta^3}{3!} + (-1)^2\frac{\theta^4}{4!} + \cdots$$

$$= 1 + i\theta + i^2\frac{\theta^2}{2!} + i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + \cdots$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots = e^{i\theta}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Complex number review

- Use Taylor series to rewrite $\cos\theta + i\sin\theta$.

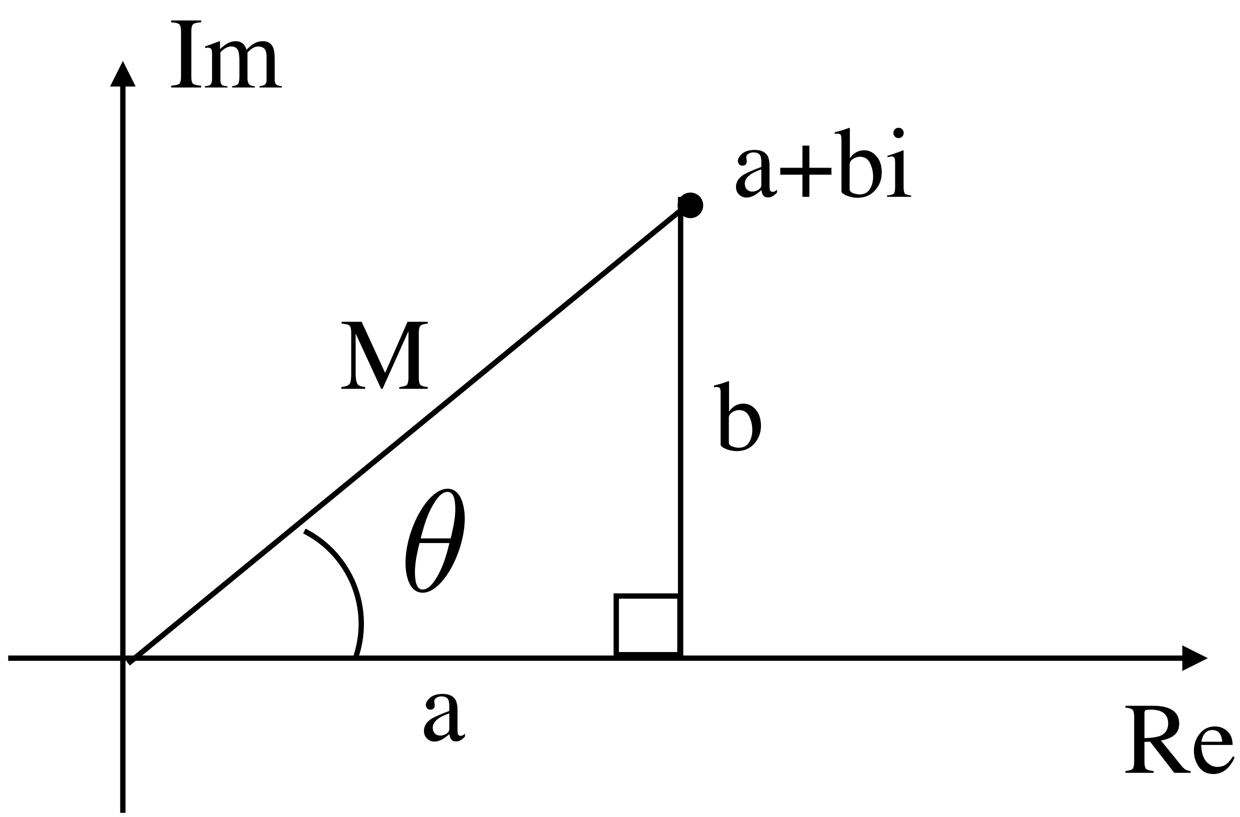
$\cos\theta + i\sin\theta$

Euler's formula:

$$= e^{i\theta}$$

Complex number review

- Geometric interpretation of complex numbers
  - e.g. $a + bi$



$$a = M\cos\theta$$

$$b = M\sin\theta$$

$$M = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

$$a + bi = M(\cos\theta + i\sin\theta)$$

$$a + bi = Me^{i\theta}$$

(Polar form makes multiplication much cleaner)

Complex roots

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- When $b^2 - 4ac < 0$, we get complex roots:

$$\begin{aligned} r_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b \pm \sqrt{-1}\sqrt{4ac - b^2}}{2a} \\ &= \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i \\ &= \alpha \pm \beta i \end{aligned}$$

Complex roots

- Complex roots to the characteristic equation mean complex valued solution to the ODE:

$$\begin{aligned} y_1(t) &= e^{(\alpha+\beta i)t} \\ &= e^{\alpha t} e^{i\beta t} \\ &= e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) \end{aligned}$$

$$\begin{aligned} y_2(t) &= e^{(\alpha-\beta i)t} \\ &= e^{\alpha t} e^{-i\beta t} \\ &= e^{\alpha t}(\cos(-\beta t) + i\sin(-\beta t)) \\ &= e^{\alpha t}(\cos(\beta t) - i\sin(\beta t)) \end{aligned}$$

Complex roots

- Complex roots to the characteristic equation mean complex valued solution to the ODE:

$$y_1(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

$$y_2(t) = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

- Instead of using these to form the general solution, let's use them to find two real valued solutions:

$$\frac{1}{2}y_1(t) + \frac{1}{2}y_2(t) = e^{\alpha t}\cos(\beta t)$$

$$\frac{1}{2i}y_1(t) - \frac{1}{2i}y_2(t) = e^{\alpha t}\sin(\beta t)$$

- General solution:

$$\boxed{y(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)}$$

Complex roots

- To be sure this is a general solution, we must check the Wronskian:

$$W(e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t))(t) =$$

(for you to fill in later - is it non-zero?)

Recall: $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

Complex roots

- Example: Find the (real valued) general solution to the equation

$$y'' + 2y' + 5y = 0$$

- Step 1: Assume $y(t) = e^{rt}$, plug this into the equation and find values of r that make it work.

(A) $r_1 = 1 + 2i,\ r_2 = 1 - 2i$

(B) $r_1 = -1 + 2i,\ r_2 = -1 - 2i$

(C) $r_1 = 1 - 2i,\ r_2 = -1 + 2i$

(A) $r_1 = 2 + 4i,\ r_2 = 2 - 4i$

(B) $r_1 = -2 + 4i,\ r_2 = -2 - 4i$

Complex roots

- Example: Find the (real valued) general solution to the equation

$$y'' + 2y' + 5y = 0$$

- Step 2: Real part of r goes in the exponent, imaginary part goes in the trig functions.

☆ (A) $y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(B) $y(t) = C_1 e^{(-1+2i)t} + C_2 e^{(-1-2i)t}$

(C) $y(t) = C_1 \cos(2t) + C_2 \sin(2t) + C_3 e^{-t}$

(D) $y(t) = C_1 \cos(2t) + C_2 \sin(2t)$

Complex roots

- Example: Find the solution to the IVP

$$y'' + 2y' + 5y = 0, \;\; y(0) = 1, \;\; y'(0) = 0$$

- General solution: $y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

(A) $y(t) = e^{-t}\left(2\cos(2t) + \sin(2t)\right)$

(B) $y(t) = e^{-t}\left(\cos(2t) - \frac{1}{2}\sin(2t)\right)$

(C) $y(t) = \frac{1}{2}e^{-t}(2\cos(2t) - \sin(2t))$

(D) $y(t) = \frac{1}{2}e^{-t}(2\cos(2t) + \sin(2t))$

Repeated roots

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- There are three cases.

  I. Two distinct real roots: $b^2 - 4ac > 0$. ($r_1 \neq r_2$)

  II. A repeated real root: $b^2 - 4ac = 0$.

  III. Two complex roots: $b^2 - 4ac < 0$.

- For case ii ($r_1 = r_2 = r$), we need another independent solution!
- Reduction of order - a method for guessing another solution.

Reduction of order

- You have one solution $y_1(t)$ and you want to find another independent one, $y_2(t)$.
- Guess that $y_2(t) = v(t)y_1(t)$ for some as yet unknown $v(t)$. If you can find $v(t)$ this way, great. If not, gotta try something else.
- Example - $y'' + 4y' + 4y = 0$. Only one root to the characteristic equation, r=-2, so we only get one solution that way: $y_1(t) = e^{-2t}$.
- Use Reduction of order to find a second solution.

$$y_2(t) = v(t)e^{-2t}$$

- Heuristic explanation for exponential solutions and Reduction of order.

Reduction of order

For the equation $y'' + 4y' + 4y = 0$, say you know $y_1(t) = e^{-2t}$.

Guess $y_2(t) = v(t)e^{-2t}$. $\quad y_2'(t) = v'(t)e^{-2t} - 2v(t)e^{-2t}$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$4y_2(t) = \cancel{4v(t)e^{-2t}} \qquad 4y_2'(t) = \cancel{4v'(t)e^{-2t}} - \cancel{8v(t)e^{-2t}}$$

$$y_2''(t) = v''(t)e^{-2t} - 2v'(t)e^{-2t} - 2v'(t)e^{-2t} + 4v(t)e^{-2t}$$

$$\searrow \quad y_2''(t) = v''(t)e^{-2t} - \cancel{4v'(t)e^{-2t}} + \cancel{4v(t)e^{-2t}}$$

$$0 = y_2'' + 4y_2' + 4y_2 = v''e^{-2t}$$

$$v'' = 0 \quad \Rightarrow \quad v' = C_1 \quad \Rightarrow \quad v(t) = C_1 t + C_2$$

Reduction of order

For the equation $y'' + 4y' + 4y = 0$, say you know $y_1(t) = e^{-2t}$.

Guess $y_2(t) = v(t)e^{-2t}$ (where $v(t) = C_1 t + C_2$).

$$= (C_1 t + C_2)e^{-2t}$$

$$y(t) = C_1 \underbrace{te^{-2t}}_{y_2(t)} + C_2 \underbrace{e^{-2t}}_{y_1(t)}$$

Is this the general solution? Calculate the Wronskian:

$$W(e^{-2t}, te^{-2t})(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^{-4t} \neq 0$$

So yes!

Summary

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- There are three cases.

I. Two distinct real roots: $b^2 - 4ac > 0$. (r_1, r_2)

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

II. A repeated real root: $b^2 - 4ac = 0$. (r)

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

III. Two complex roots: $b^2 - 4ac < 0$. ($r_{1,2} = \alpha \pm i\beta$)

$$y = e^{\alpha t} \left(C_1 \cos(\beta t) + C_2 \sin(\beta t) \right)$$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 8y = 0$$

(A) $y(t) = C_1 e^{-2t} + C_2 e^{-4t}$

(B) $y(t) = C_1 e^{2t} + C_2 e^{4t}$

(C) $y(t) = e^{2t}(C_1 \cos(4t) + C_2 \sin(4t))$

(D) $y(t) = e^{-2t}(C_1 \cos(4t) + C_2 \sin(4t))$

(E) $y(t) = C_1 e^{2t} + C_2 t e^{4t}$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 9y = 0$$

(A) $y(t) = C_1 e^{3t}$

(B) $y(t) = C_1 e^{3t} + C_2 e^{3t}$

(C) $y(t) = C_1 e^{3t} + C_2 e^{-3t}$

☆ (D) $y(t) = C_1 e^{3t} + C_2 t e^{3t}$

(E) $y(t) = C_1 e^{3t} + C_2 v(t) e^{3t}$

Second order, linear, constant coeff, homogeneous

- Find the general solution to the equation

$$y'' - 6y' + 10y = 0$$

(A) $y(t) = C_1 e^{3t} + C_2 e^{t}$

(B) $y(t) = C_1 e^{3t} + C_2 e^{-t}$

(C) $y(t) = C_1 \cos(3t) + C_2 \sin(3t)$

(D) $y(t) = e^{t}(C_1 \cos(3t) + C_2 \sin(3t))$

(E) $y(t) = e^{3t}(C_1 \cos(t) + C_2 \sin(t))$