Today

- Shapes of solutions for distinct eigenvalues case.

Example

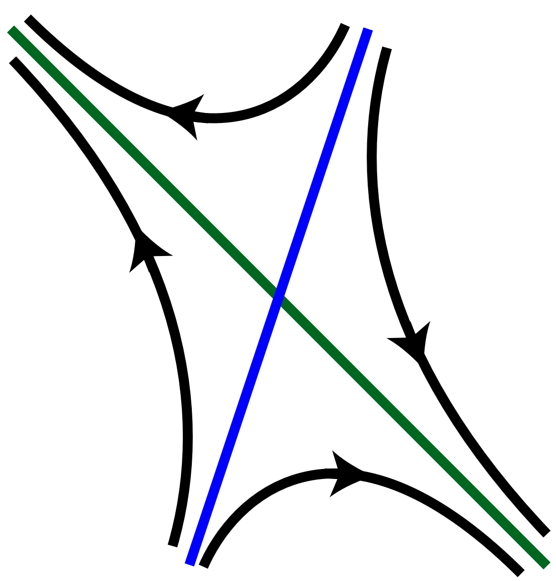
- Doc cam:
- $y(t) = C_1(1; 2)\, e^{-t} + C_2(1; -1)\, e^{t}$
  - With ICs
    - $y(0) = (2;4)$
    - $y(0) = (2;2)$
    - $y(0) = (2;1)$
  - Desmos: https://www.desmos.com/calculator/tpelfq4nbe

Shapes of solution curves in the phase plane

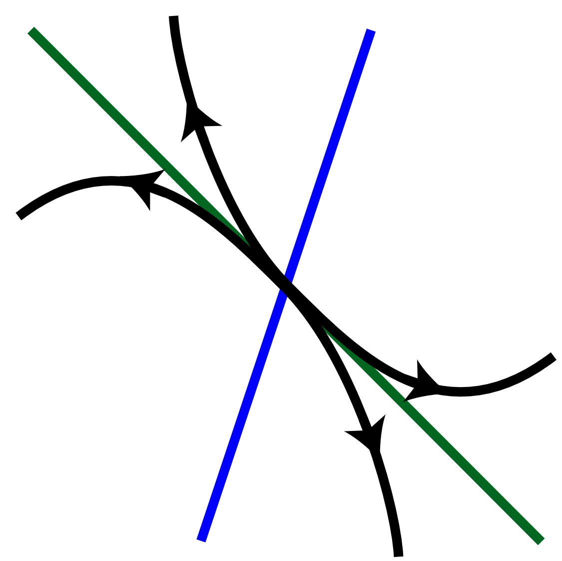
- Which phase plane matches the general solution

$$\mathbf{x} = C_1 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ ?}$$

(A)



(B)



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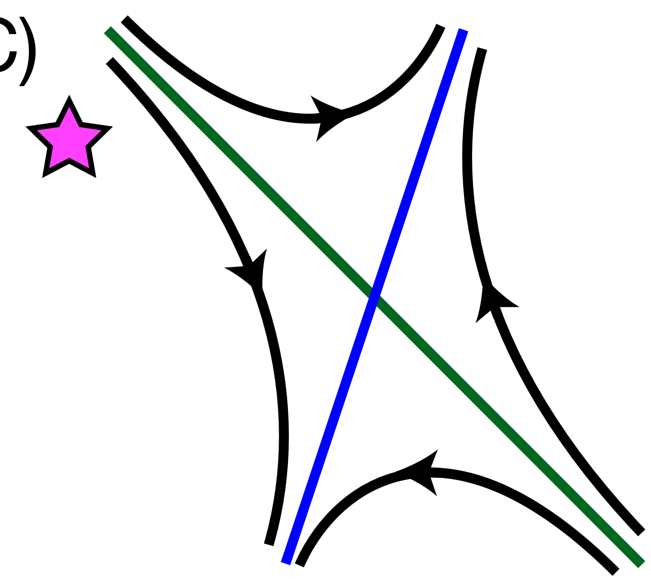
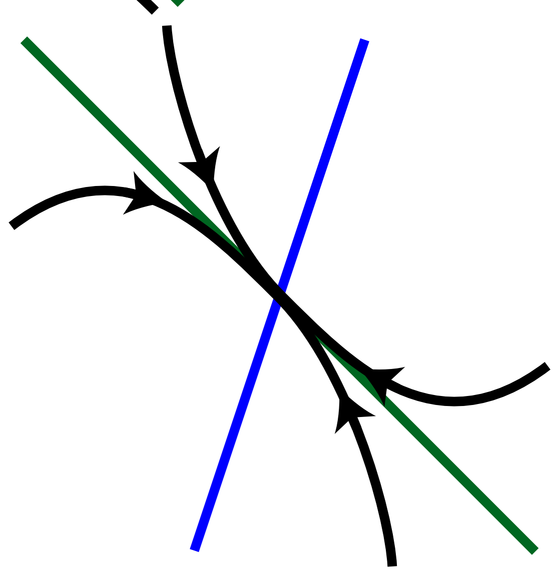
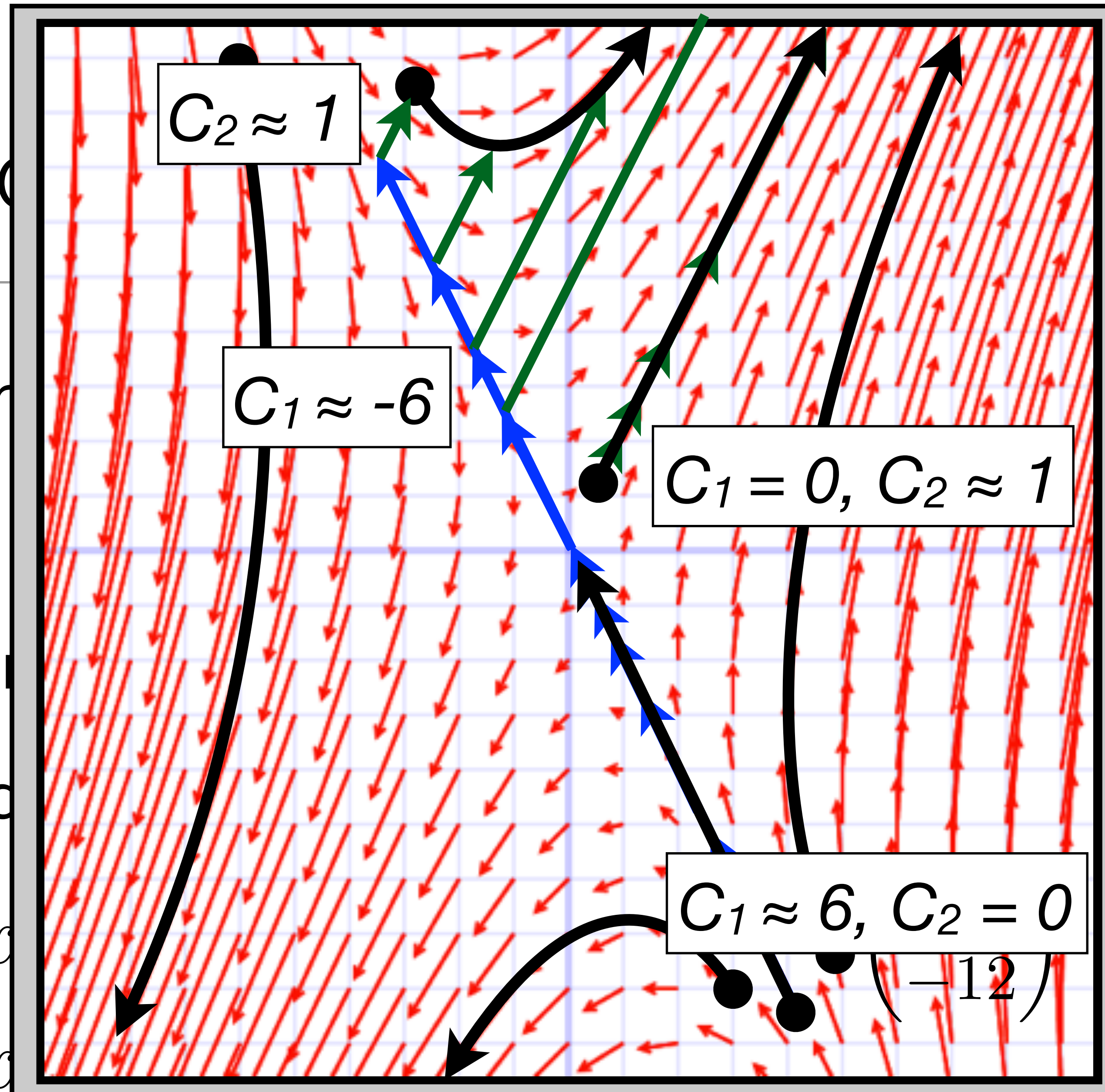


(D)



(E) Explain, please.

Shapes of so… plane



- When matrix A h… olution to **x**'=A**x** is
- What do solutio… e phase plane)?
- If the initial cond… on is a straight line. Example:

x … 6 … $\begin{pmatrix} \\ -12 \end{pmatrix}$

x … -12

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boxed{C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}} + \boxed{C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

$$C_1 = 6, \; C_2 = 0$$

- IVP solution:

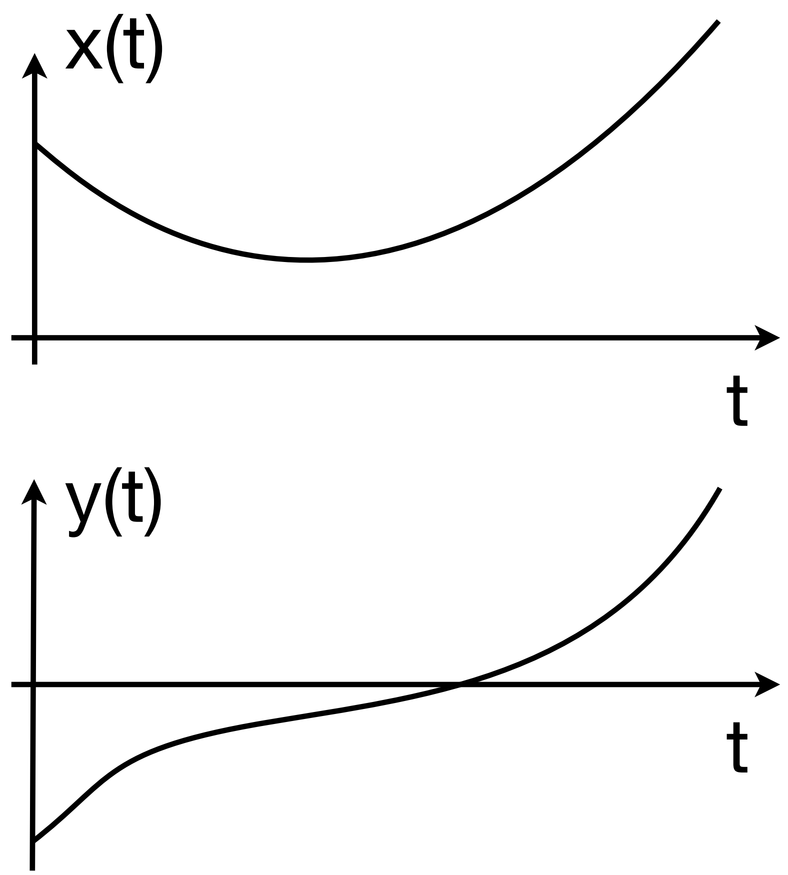
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

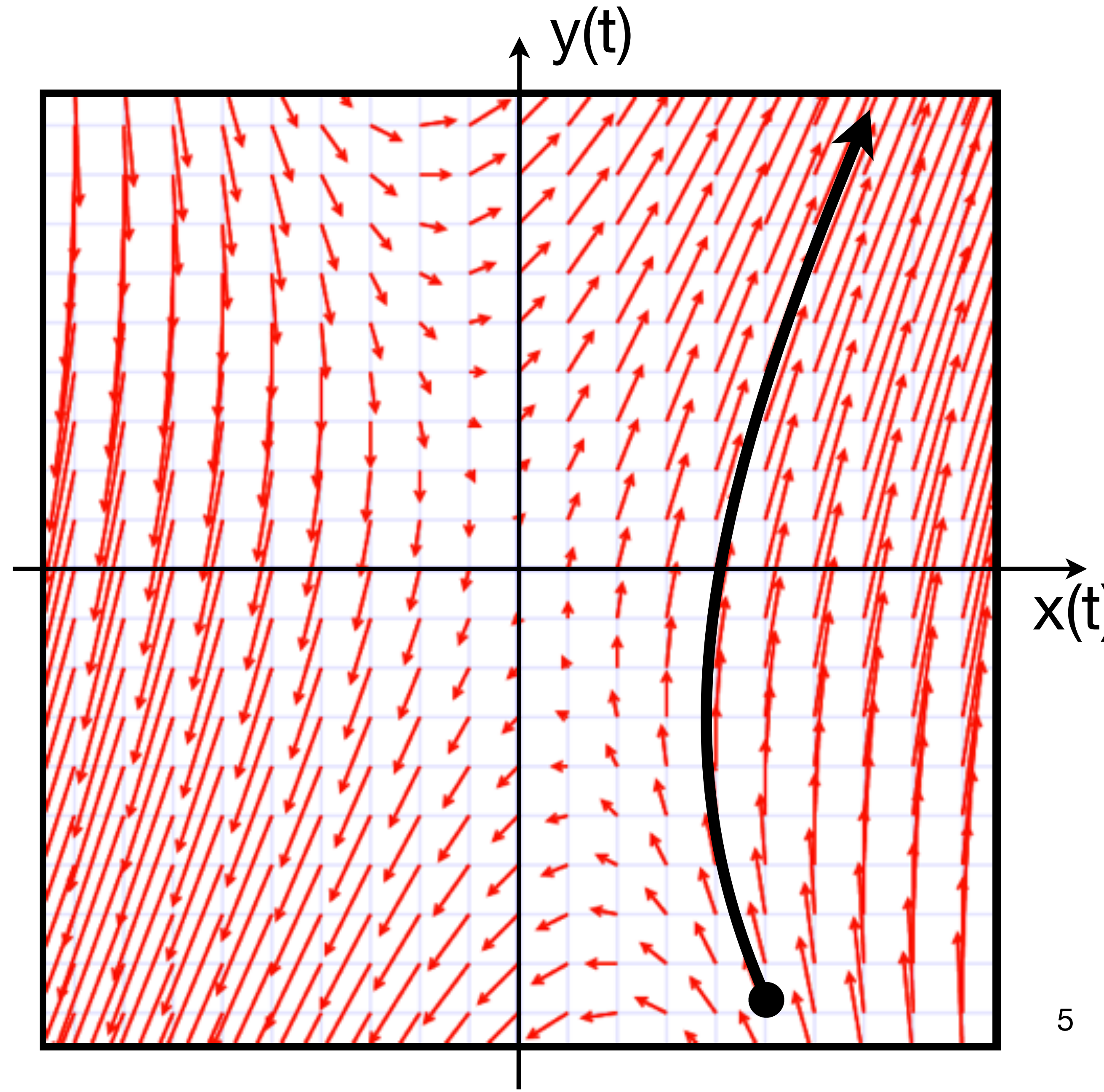
Plotting x(t) vs y(t) compared to t vs x(t)

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

$$C_1 = \frac{7}{2},\ C_2 = \frac{1}{2}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{7}{2} e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{1}{2} e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$





Shapes of solution curves in the phase plane

- Simple example to show general idea. $\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{x}$

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = C_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_1(t) = C_1 e^{\lambda_1 t} \qquad t = \frac{1}{\lambda_1} \ln\left(\frac{x_1}{C_1}\right)$$

$$x_2(t) = C_2 e^{\lambda_2 t} \qquad t = \frac{1}{\lambda_2} \ln\left(\frac{x_2}{C_2}\right)$$

$$\frac{1}{\lambda_2} \ln\left(\frac{x_2}{C_2}\right) = \frac{1}{\lambda_1} \ln\left(\frac{x_1}{C_1}\right)$$

$$\ln\left(\frac{x_2}{C_2}\right) = \frac{\lambda_2}{\lambda_1} \ln\left(\frac{x_1}{C_1}\right)$$

$$\ln\left(\frac{x_2}{C_2}\right) = \ln\left(\frac{x_1}{C_1}\right)^{\frac{\lambda_2}{\lambda_1}}$$

$$x_2 = C_2 \left(\frac{x_1}{C_1}\right)^{\frac{\lambda_2}{\lambda_1}}$$

- Can we plot solutions in x_1-x_2 plane by graphing x_2 versus x_1?

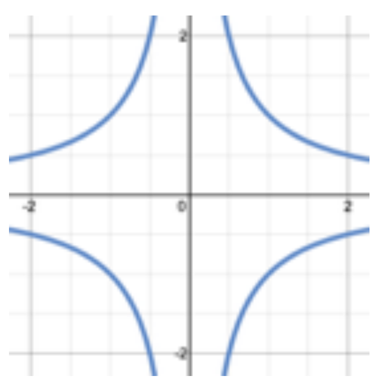
Shapes of solution curves in the phase plane

- Simple example to show general idea. $\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{x}$

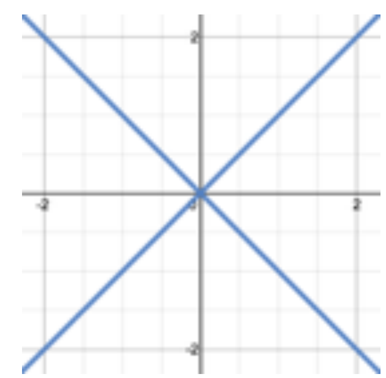
$$x_2 = C_2 \left(\frac{x_1}{C_1} \right)^{\frac{\lambda_2}{\lambda_1}}$$

- For the shape of solutions, we need to know the sign and size of $\dfrac{\lambda_2}{\lambda_1}$.

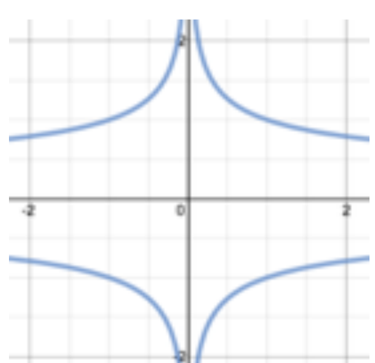
$$\lambda_2 = -\lambda_1$$
$$x_2 = \frac{C}{x_1}$$



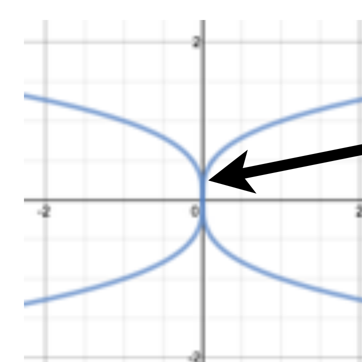
$$\lambda_2 = \lambda_1$$
$$x_2 = C x_1$$



$$\lambda_2 = -\frac{1}{3}\lambda_1$$
$$x_2 = \frac{C}{\sqrt[3]{x_1}}$$



$$\lambda_2 = \frac{1}{3}\lambda_1$$
$$x_2 = C\sqrt[3]{x_1}$$

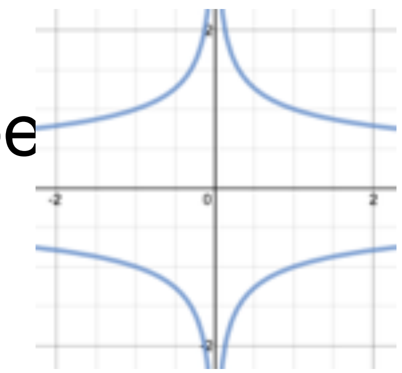


stays near x_2 axis

https://www.desmos.com/calculator/c4rhrgotmo

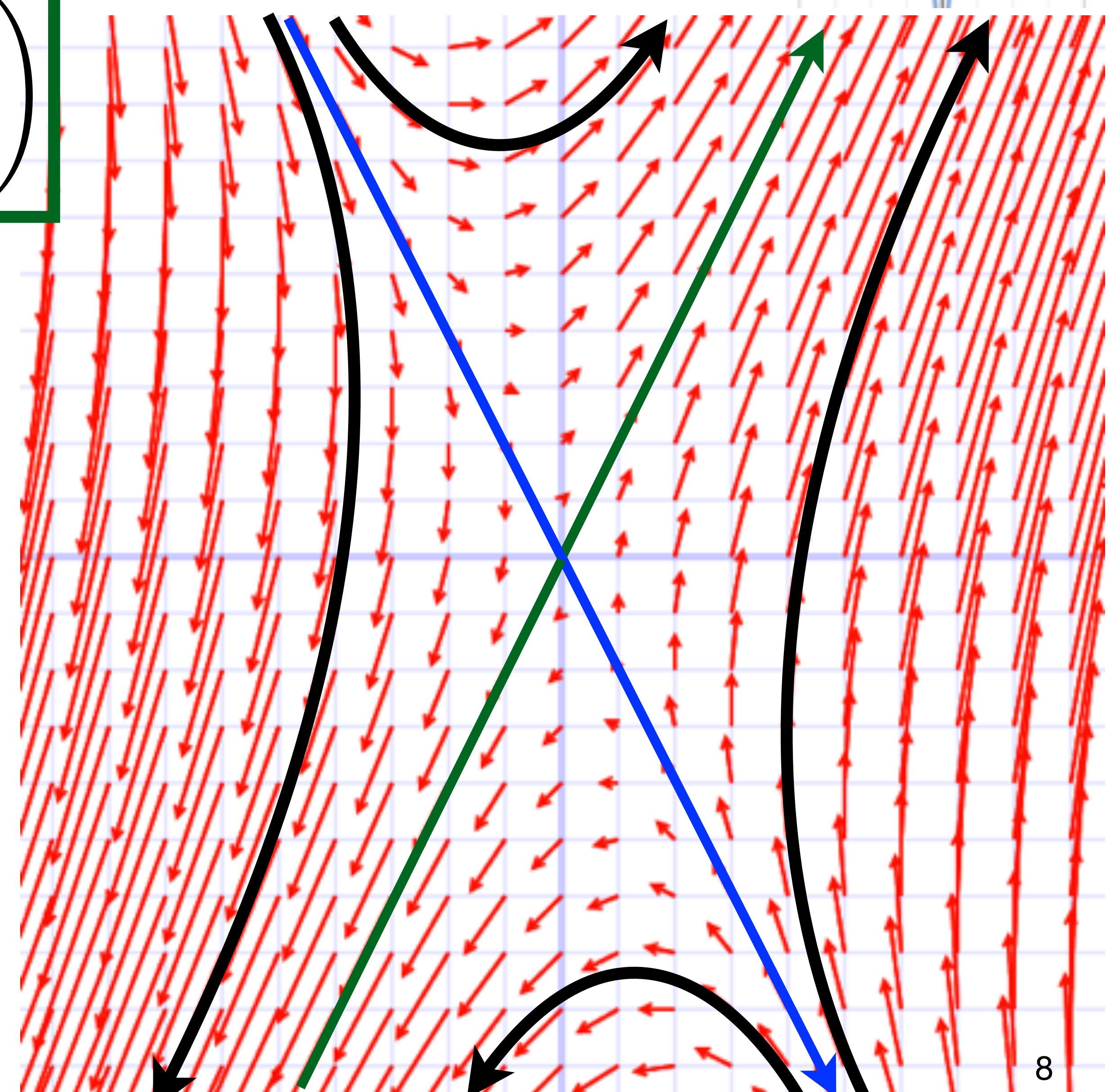
Shapes of solution curves in the phase plane

- With more complicated solutions (evectors off-axis), tilt shape accordingly.



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boxed{C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}} + \boxed{C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

- Going forward in time, the blue component shrinks slower than the green component grows so solutions appear closer to blue "axis" than to green "axis"



Shapes of solution curves in the phase plane

- Which phase plane matches the general solution

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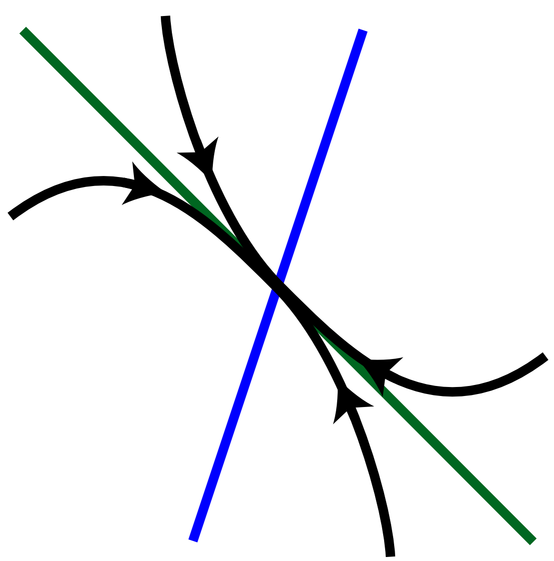
(A)



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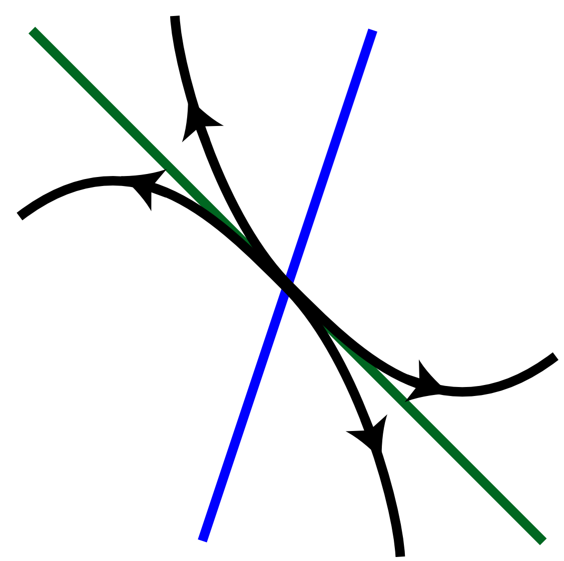
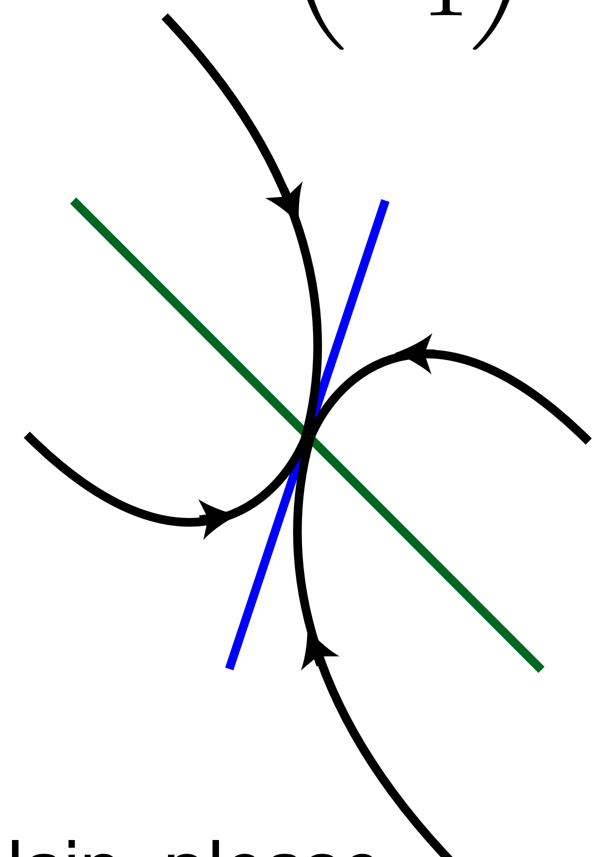
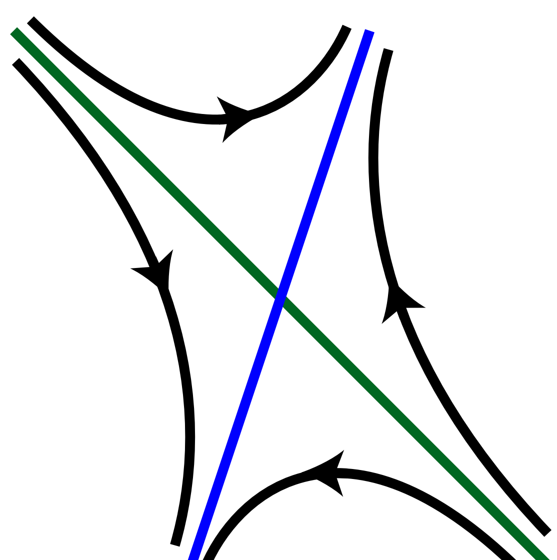


(C)



(D)



(E) Explain, please.