Today

- Reminders:
  - Pre-lecture assignment for Thursday 7 am
  - Week 1 assignment due Friday 5 pm.
- Separating variables
- Modeling tank inflow/outflow scenarios
- Existence and uniqueness (not going to test on the theory but important to know for general understanding)

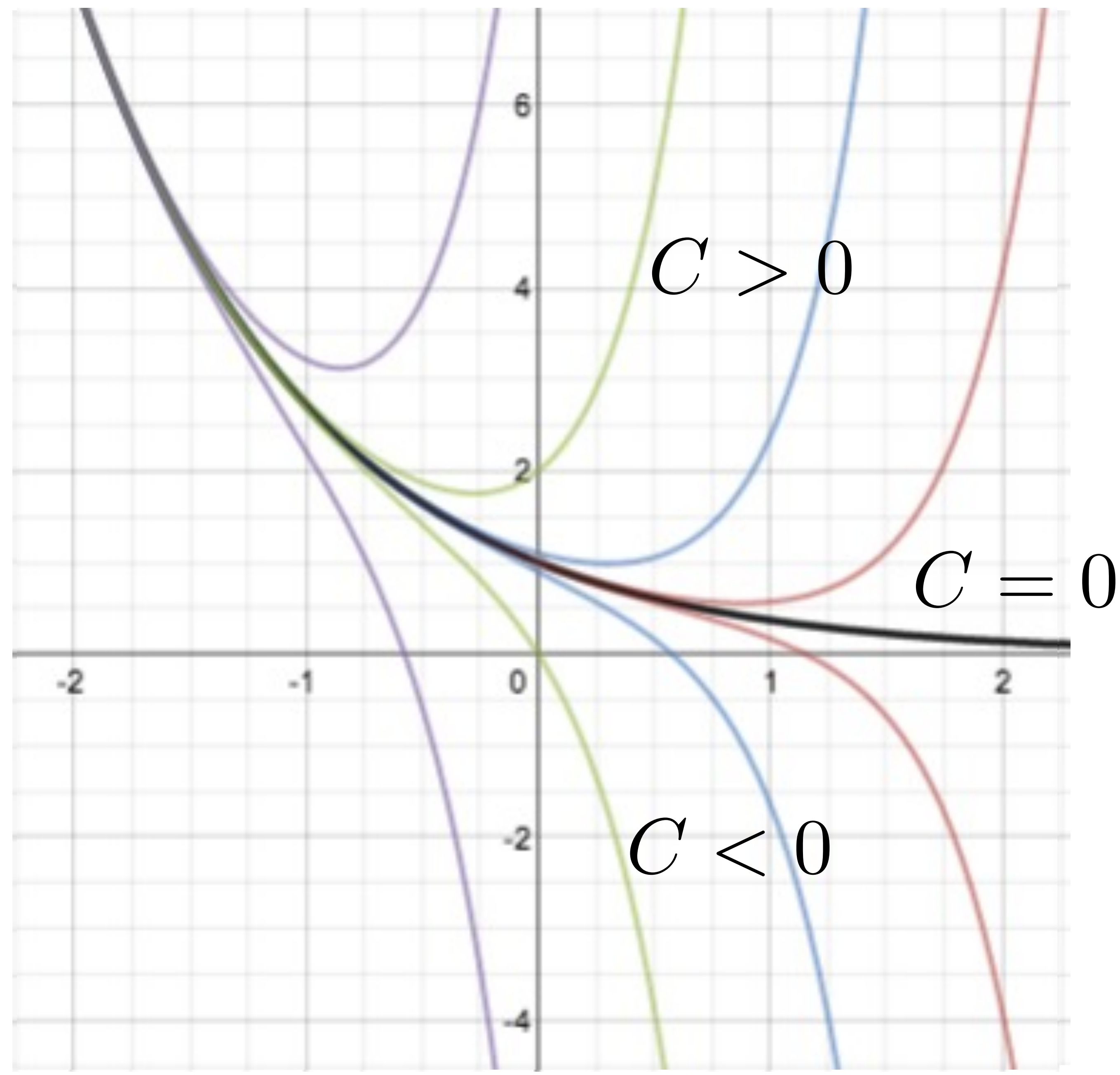
Examples

- Find the general solution to

$$\frac{dy}{dt} - 3y = -4e^{-t}$$

- and plot a few of the integral curves.

$$y(t) = e^{-t} + Ce^{3t}$$



Limits at infinity

- If y(t) is a particular solution to

$$\frac{dy}{dt} - 3y = -4e^{-t}$$

- depending on C, how many different results are possible for

$$\lim_{t\to\infty} y(t) \text{ ?}$$

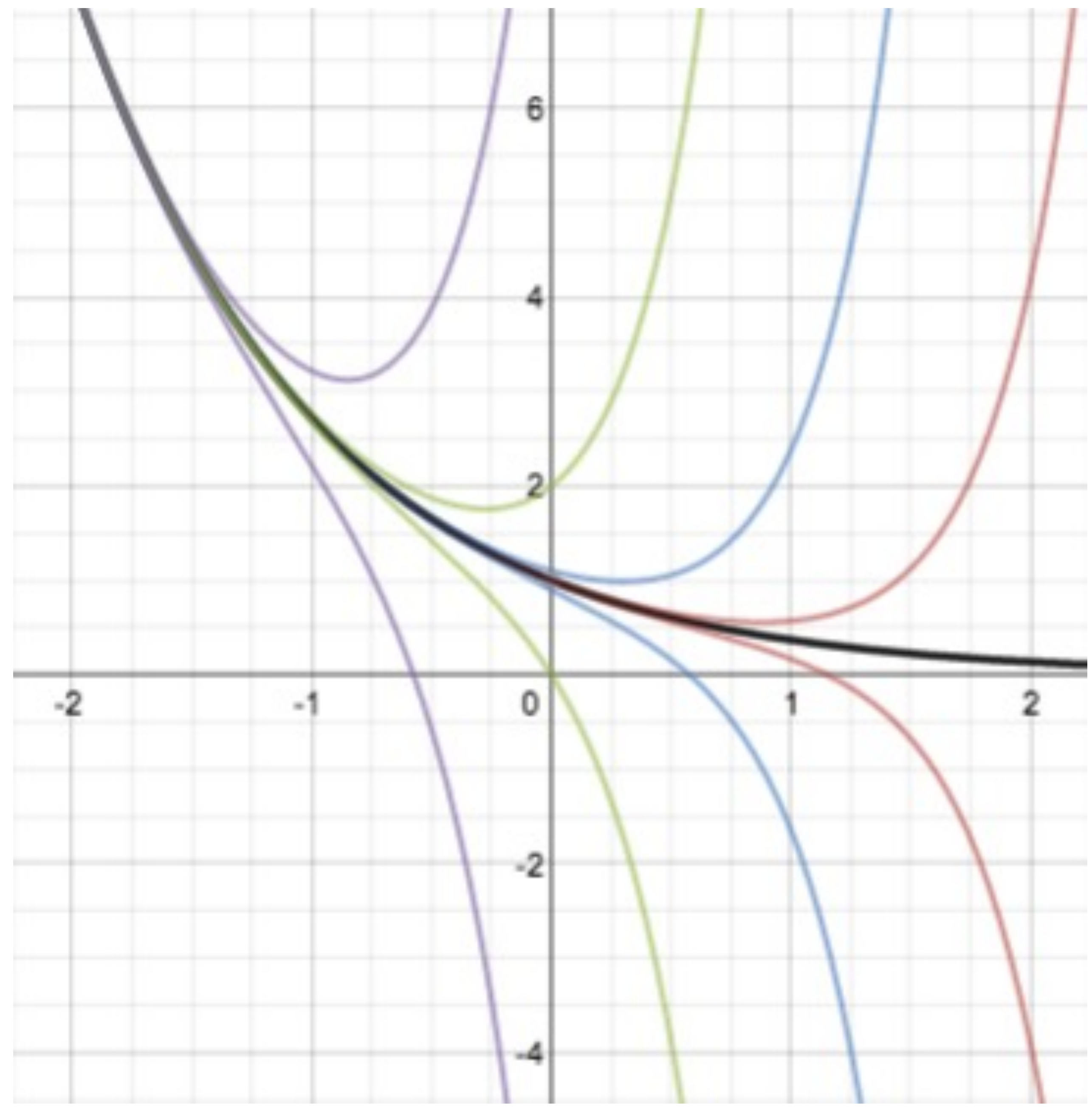
(A) 0

(B) 1

(C) 2

(D) 3

(E) Don’t know.



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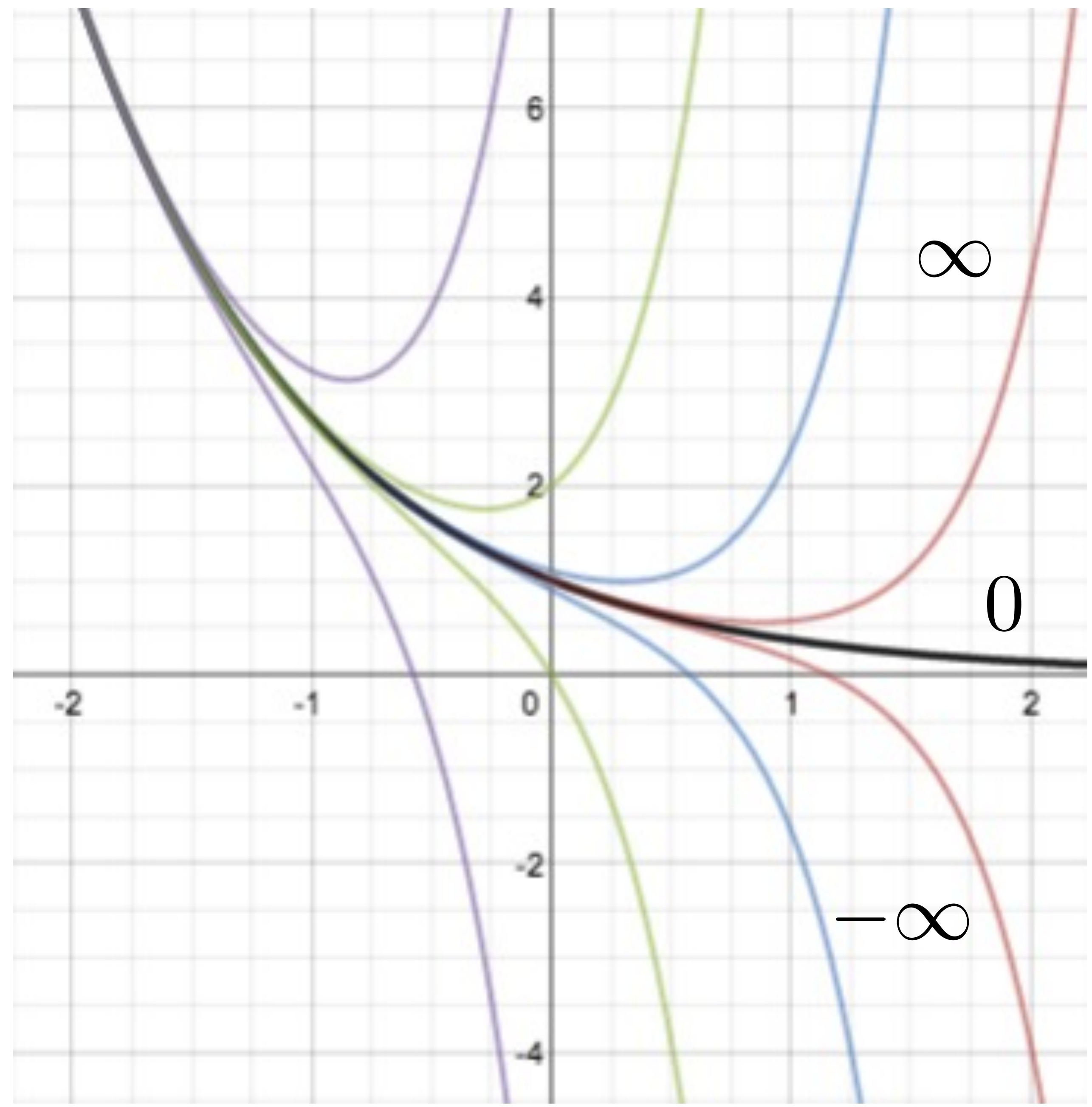


Limits at infinity

- If y(t) is a particular solution to

$$\frac{dy}{dt} - 3y = -e^t$$

- depending on C, how many different results are possible for

$$\lim_{t\to\infty} y(t) \text{ ?}$$

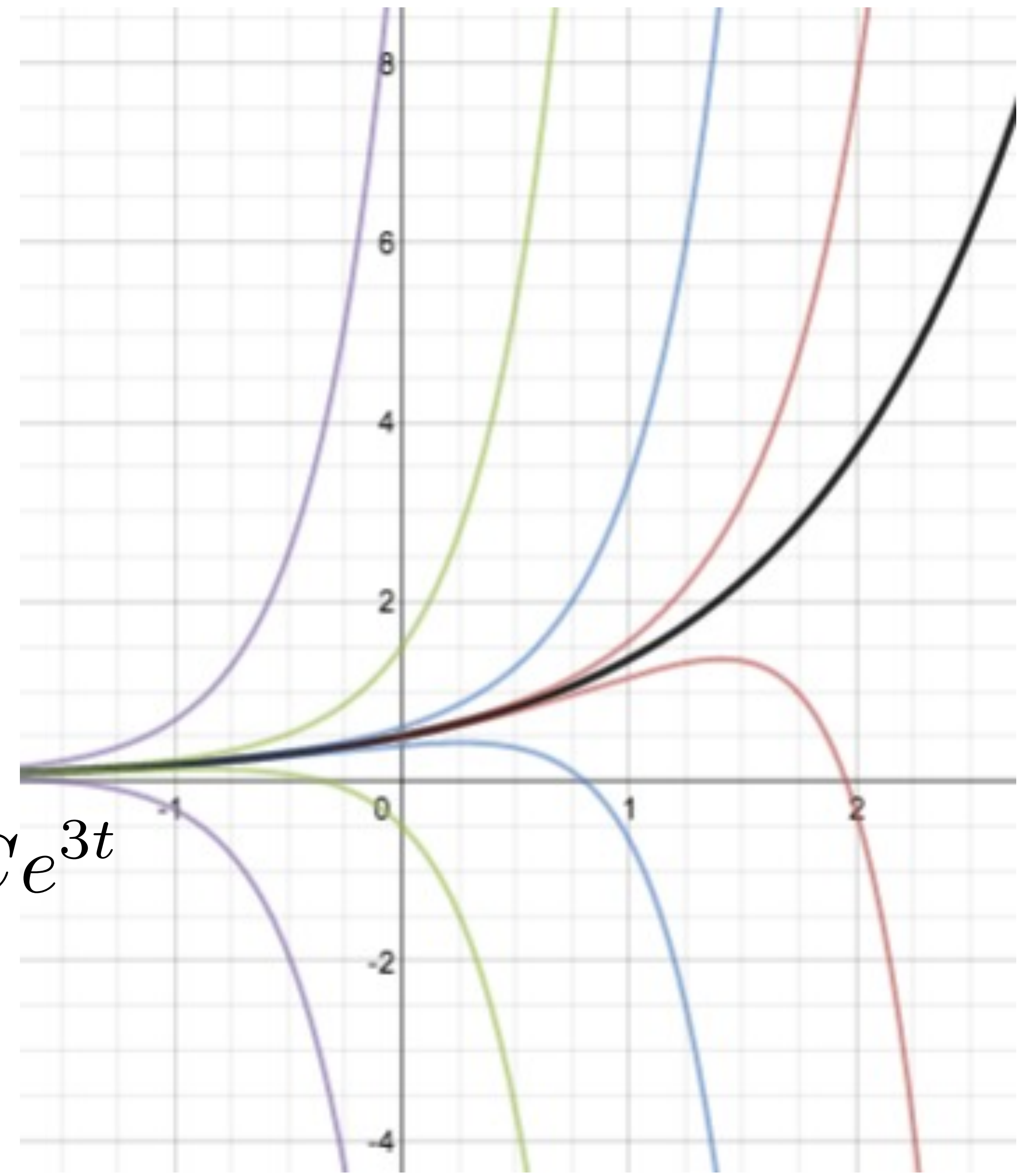
(A) 0

(B) 1

(C) 2

(D) 3

$$y(t) = \frac{1}{2}e^t + Ce^{3t}$$



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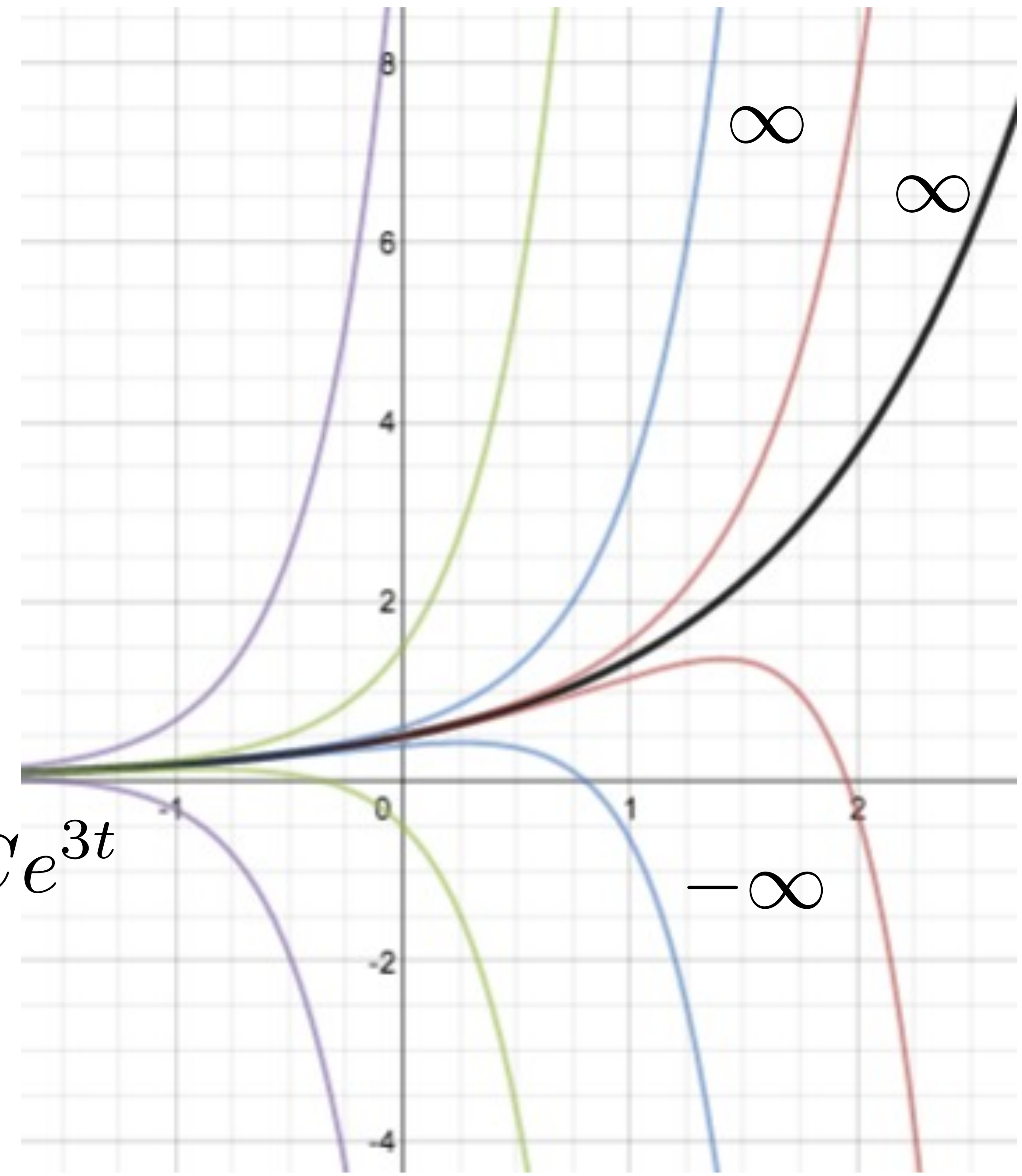

(E) Don’t know.

Separable equations

- What is $\dfrac{d}{dt}e^y$?

(A) e^y

(B) $e^y \dfrac{dy}{dt}$

(C) ye^{y-1}

(D) $ye^{y-1}\dfrac{dy}{dt}$

(E) Don’t know.

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Separable equations

- What is $\frac{d}{dt}e^y$?

Hint: rewrite as $e^y \frac{dy}{dt} = t^2$.

$$\frac{d}{dt}(e^y) = t^2$$

$$e^y = \frac{1}{3}t^3 + C$$

(D) ye dt

(E) Don't know.

- Solve $\frac{dy}{dt} = e^{-y}t^2$.

(A) $y(t) = t^2e^t + C$

(B) $y(t) = \frac{1}{3}t^3 + C$

(C) $y(t) = \ln\left(\frac{1}{3}t^3\right) + C$

(D) $y(t) = \ln\left(\frac{1}{3}t^3 + C\right)$

(E) Don't know.

Separable equations

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Hint: rewrite as $e^y \dfrac{dy}{dt} = t^2$.

$$\frac{d}{dt}(e^y) = t^2$$

$$e^y = \frac{1}{3}t^3 + C$$

(D) ye … $\dfrac{\quad}{dt}$

(E) Don’t know.

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(D) $y(t) = \ln\left(\dfrac{1}{3}t^3 + C\right)$

(E) Don’t know.

Separable equations

- First order ODEs of the form: $\dfrac{dy}{dx} = f(x)h(y)$
- Rename h(y) = 1/g(y): $\dfrac{dy}{dx} = \dfrac{f(x)}{g(y)}$
- Rewrite as $g(y)\dfrac{dy}{dx} = f(x)$.
- Rewrite g and f as derivatives of other functions: $G'(y)\dfrac{dy}{dx} = F'(x)$.
- Recognize a chain rule: $\dfrac{d}{dx}(G(y)) = G'(y)\dfrac{dy}{dx}$.
- Take antiderivatives to get $G(y) = F(x) + C$.
- Finally, solve for y if possible: $y(x) = G^{-1}(F(x) + C)$.

Separable equations

- Solve: $$\frac{dy}{dx} = -\frac{x}{y}$$

(A) $y(x) = x$

(B) $y(x) = \sqrt{C - x^2}$

(C) $y(x) = \sqrt{x^2 + C}$

(D) $y(x) = C - x^2$

(E) None of these (or don’t know)

$$y\frac{dy}{dx} = -x$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + D$$

$$y^2 = -x^2 + C$$

Does (B) cover all possible initial conditions?

Separable equations

$$y(x) = \sqrt{C - x^2}$$

- y(0)=2 ----> C=4
- y(1)=1 ----> C=2
- y(1)=-2 ----> C=?
- General solution: $y = \pm\sqrt{C - x^2}$
- Or express implicitly: $y^2 = -x^2 + C$
- To satisfy an IC, must choose a value for C *and* choose + or - .

Separable equations

- Solve: $$\frac{dy}{dt} = \frac{1}{\cos(y)}$$

(A) $y(t) = \sin(t)$

(B) $y(t) = \arcsin(t + C)$

(C) $\sin(y) = t + C$

(D) $y(t) = \arcsin(t) + C$

(E) $y(t) = \arccos(t + C)$

Separable equations

$\sin(y) = t + C$

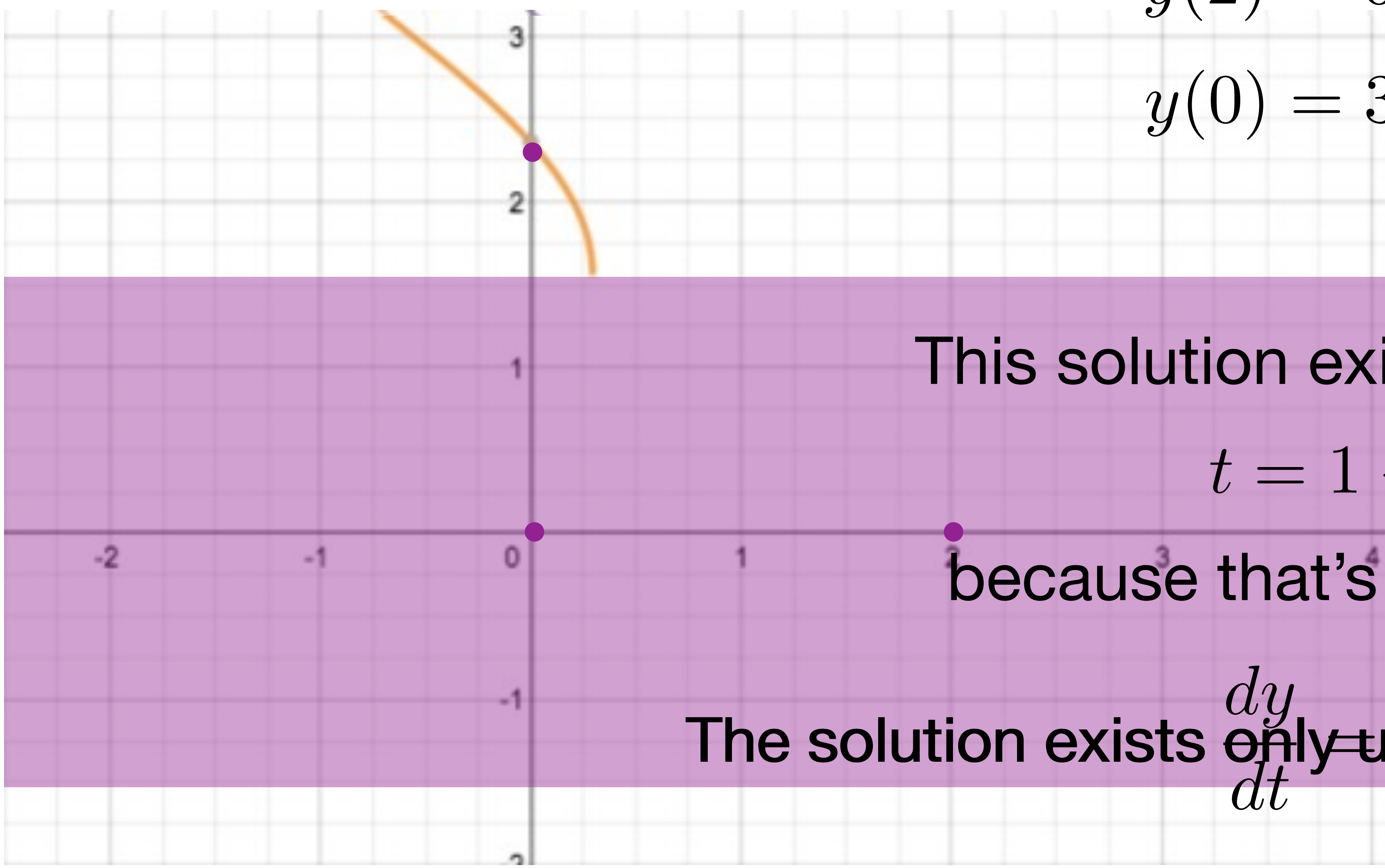
$y(t) = \arcsin(t + C)$

with IC

$y(0) = 0 \qquad C = 0$

$y(2) = 0 \qquad C = -2$

$y(0) = 3\pi/4 \quad C = \dfrac{1}{\sqrt{2}}$



This solution exists only up until

$$t = 1 - 1/\sqrt{2}$$

because that's when y = π/2.

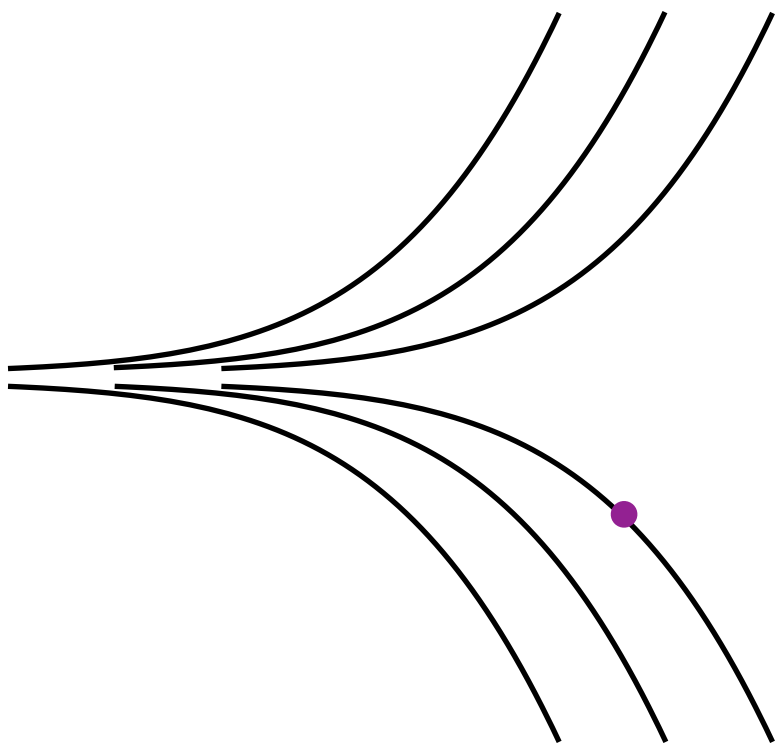
$$\frac{dy}{dt} = \frac{1}{\cos(y)}$$

The solution exists only up until t=1.

The solution exists only up until t=3.

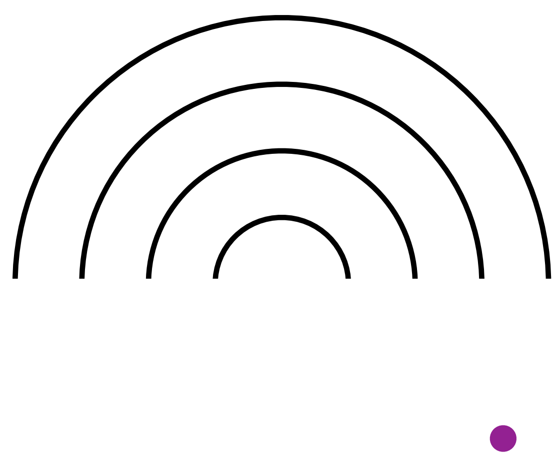
When do we get this kind of problem with ICs?

$$y = Ce^{3t}$$



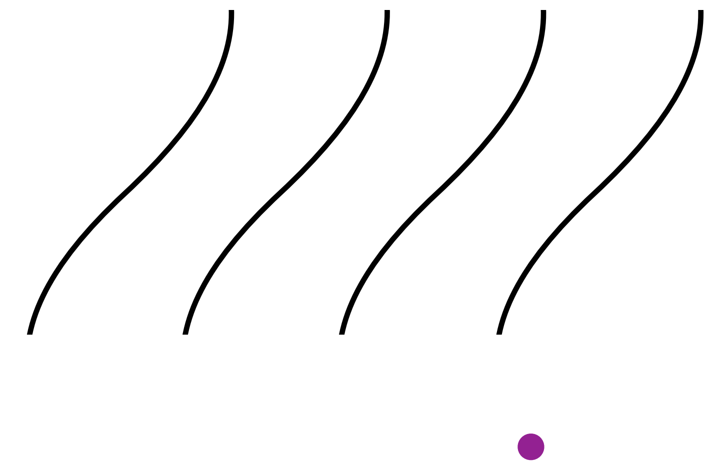
$$y^2 = -x^2 + C$$

$$y(x) = \sqrt{C - x^2}$$



$$\sin(y) = t + C$$

$$y(t) = \arcsin(t + C)$$



Choose C so that the function goes through the IC.

Modeling tanks with solution flowing in/out

- Inflow/outflow problems
    - Determine what quantity(-ies) to track (e.g. mass, concentration, temperature, etc.).
    - Choose a small interval of time, Δt, and add up all the changes.
    - Note that $q(t + \Delta t) = q(t) +$ change during intervening Δt.
    - More specifically, $q(t + \Delta t) \approx q(t) \ + (\text{inflow rate - outflow rate})\Delta t$
    - Rearrange and take limit as $\Delta t \to 0$ to get an equation for q(t).

Modeling - Example

- Freshwater flows into a tank at a rate 2 L/min. The tank starts with a concentration of 100 g / L of salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

(a) Write down an IVP for the mass of salt in the tank as a function of time.

(b) What is the limiting mass of salt in the tank ($\lim_{t\to\infty} m(t)$)?

(a) What is the change in the mass of salt in any short interval of time Δt?

(A) $\Delta m \approx -2\ \mathrm{L/min}\ \times\ m(t)\ /\ 10\ \mathrm{L}$

(B) $\Delta m \approx -2\ \mathrm{L/min}\ \times 100\ \mathrm{g/L} \times \Delta t$

★ (C) $\Delta m \approx -2\ \mathrm{L/min}\ \times\ m(t)\ /\ 10\ \mathrm{L}\ \times\ \Delta t$

(D) $\Delta m \approx -2\ \mathrm{L/min}\ \times\ m(t)$

Modeling - Example

- Freshwater flows into a tank at a rate 2 L/min. The tank starts with a concentration of 100 g / L of salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank ($\lim_{t\to\infty} m(t)$)?

- $m(t+\Delta t) = m(t) + \Delta m$ so
- $m(t+\Delta t) \approx m(t) - \Delta t \ \times \ 2 \text{ L/min} \ \times \ m(t) \ / \ 10 \text{ L}$
- Rearranging: $\dfrac{m(t+\Delta t) - m(t)}{\Delta t} \approx -\dfrac{1}{5} m(t)$
- Finally, taking a limit:

$$\boxed{\frac{dm}{dt} = -\frac{1}{5} m(t)}$$

Modeling - Example

- Freshwater flows into a tank at a rate 2 L/min. The tank starts with a concentration of 100 g / L of salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

    (a) Write down an IVP for the mass of salt in the tank as a function of time.

    (b) What is the limiting mass of salt in the tank ($\lim_{t\to\infty} m(t)$)?

(a) We got the equation (m'=-1/5 m). Now what is the initial condition?

- m(0) = 1000 g.

Modeling - Example

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- What method could you use to solve the ODE $\dfrac{dm}{dt} = -\dfrac{1}{5}m(t)$?

  (A) Integrating factors.

  (B) Separating variables.

  (C) Just knowing some derivatives.

  ☆ (D) All of these.

  (E) None of these.

To think about: what is the most general equation that can be solved using (A) and (B)?

Modeling - Example

- Freshwater flows into a tank at a rate 2 L/min. The tank starts with a concentration of 100 g / L of salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank ($\lim_{t \to \infty} m(t)$)?

- The solution to the IVP is

  (A) $m(t) = Ce^{-t/5}$

  (B) $m(t) = 100e^{-t/5}$

  (C) $m(t) = 100e^{t/5}$

  (D) $m(t) = 1000e^{t/5}$

  ☆ (E) $m(t) = 1000e^{-t/5}$

Answer to (b)?

$$\lim_{t \to \infty} m(t) = 0$$

Modeling - Example

- Saltwater with a concentration of 200 g/L flows into a tank at a rate 2 L/min. The tank starts with no salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

    (a) Write down an IVP for the mass of salt in the tank as a function of time.

    (b) What is the limiting mass of salt in the tank?

(a) The IVP is

(A) $m' = 200 - 2m, \quad m(0) = 0$

(B) $m' = 400 - 2m, \quad m(0) = 200$

(C) $m' = 400 - m/5, \quad m(0) = 0$

(D) $m' = 200 - m/5, \quad m(0) = 0$

(E) $m' = 400 - m/5, \quad m(0) = 200$

Modeling - Example

- Saltwater with a concentration of 200 g/L flows into a tank at a rate 2 L/min. The tank starts with no salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank?

(b) Directly from the equation ($m' = 400 - m/5$), find an m for which $m'=0$.

- $m=2000$. Called steady state - a constant solution.
- What happens when $m < 2000$? ---> $m' > 0$.
- What happens when $m > 2000$? ---> $m' < 0$.
- Limiting mass: 2000 g (Long way: solve the eq. and let $t\to\infty$.)