# MATH 256-201 – Midterm 1 – January 31, 2017.

Surname: Solutions  Given name: ____________

Tutorial TA (circle one): Thomas Jummy Sarai Colin Xiaowei Dhananjay

This midterm has 6 pages including a blank page at the end for rough work. Answers must be justified and work must be shown. If a box is provided, place your answer in it.

1. **[5 pts]** Classify each of the following equations as linear (L) or non-linear (NL). Give the order of the equation. For any linear equation, state whether it is homogeneous (H) or non-homogeneous (NH); put a "–" for non-linear equations. **For any non-linear equation, circle all terms that render it non-linear.**

| Equation | L/NL | Order | H/NH |
|---|---|---|---|
| $t^2y'' + 4ty' + 2y = 0$ | L | 2nd | H |
| $y' + 3y = \cos(2t)$ | L | 1st | NH |
| $y'' + xy' + y = \dfrac{ye^x}{x^2+1}$ | L | 2nd | H |

$$y'' + xy' + \left(1 - \frac{e^x}{x^2+1}\right)y = 0$$

1pt for each

2. **[5 pts]** A tank initially contains $m_0$ kg of salt and a volume $V$ litres of water. Saltwater with a concentration of $c_0$ kg/litre enters a tank at the rate $r$ litres/minute. The solution is mixed and drains from the tank at the same rate ($r$ litres/minute). Write down an Initial Value Problem (that is, **a differential equation** and **an initial condition**) for the mass of salt $m(t)$ in the tank as a function of time. You DO NOT need to solve it.

$$\frac{dm}{dt} = rc_0 - \frac{rm}{V}$$

$$m(0) = m_0$$

1pt for each

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3. Consider the equation

$$\frac{dy}{dx} = \frac{\sqrt{x}}{y}.$$

(a) **[4 pts]** Find the general solution to the equation.

$yy' = \sqrt{x}$

$\frac{1}{2}y^2 = \frac{2}{3}x^{3/2} + C$

$\left( y = \pm\sqrt{\frac{4}{3}x^{3/2} + D} \quad \text{also ok} \right)$

General solution: $\boxed{\frac{1}{2}y^2 = \frac{2}{3}x^{3/2} + C}$

(b) **[2 pts]** What is the particular solution that solves the initial condition $y(1) = -\frac{2}{\sqrt{3}}$?

$\frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right)^2 = \frac{2}{3}(1)^{3/2} + C$

$\frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3} + C$

$\frac{2}{3} = \frac{2}{3} + C$

$C = 0$

$y = \pm\sqrt{\frac{4}{3}x^{3/2} + 0}$

Must choose − for IC.

$y(x) = \boxed{-\sqrt{\frac{4}{3}x^{3/2}}}$

4. **[4 pts]** Find the general solution to the equation $y'' + 4y = 0$.

$y = e^{rt}$

$r^2 + 4 = 0$

$r = \pm 2i$

$y_h(x) = \boxed{C_1 \cos 2t + C_2 \sin 2t}$

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5. **[6 pts]** For each proposed $f(t)$, give the form of the particular solution that you would use to carry out the Method of Undetermined Coefficients to solve the equation $y'' + 4y = f(t)$.

| $f(t)$ | $y_p(t)$ |
|---|---|
| $t^2$ | $At^2 + Bt + C$ |
| $e^t \sin(2t)$ | $e^t(A\sin 2t + B\cos 2t)$ |
| $\cos(2t)$ | $t(A\cos 2t + B\sin 2t)$ |

6. **[6 pts]** Find the particular solution that solves the equation $y'' + 4y = 5te^t$.

$$y_p = (At+B)e^t$$
$$y_p' = (At+B)e^t + Ae^t$$
$$y_p'' = (At+B)e^t + Ae^t + Ae^t$$

$$y_p'' + 4y_p = Ate^t + (B+2A)e^t + 4Ate^t + 4Be^t$$
$$= 5Ate^t + (2A+5B)e^t = 5te^t$$

$$5A = 5 \qquad 2A+5B=0$$
$$A = 1 \qquad 2+5B=0$$
$$B = -\tfrac{2}{5}$$

$$y_p = \left(t - \tfrac{2}{5}\right)e^t$$

$y_p(x) =$ $\boxed{\left(t - \tfrac{2}{5}\right)e^t}$

Do not write in these boxes - for marking purposes only. 5: 6:

7. For each of the following pairs of functions, show that they are either dependent or independent. Note that a Wronskian of zero does not ensure dependence - you have to show that a non-trivial linear combination of the functions adds to zero.

(a) **[3 pts]** $f(t) = e^t$ and $g(t) = te^t$.

$$W(e^t, te^t) = e^t(te^t + e^t) - e^t \cdot te^t$$
$$= te^{2t} + e^{2t} - te^{2t} = e^{2t} \neq 0$$

Therefore, $e^t$ and $te^t$ are independent.

(b) **[3 pts]** $f(x) = \ln(x^2)$ and $g(x) = \ln(x^3)$.

$$f(x) = \ln(x^2) = 2\ln x$$
$$g(x) = \ln(x^3) = 3\ln x$$
$$3f(x) - 2g(x) = 3(2\ln x) - 2\cdot(3\ln x)$$
$$= 0$$

Therefore, $f(x)$ and $g(x)$ are dependent.

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8. (a) **[4 pts]** Find the general solution to the equation

$$y' - \frac{1}{x}y = \frac{1}{x}.$$

$I(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$\frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x^2}$

$\left(\frac{1}{x}y\right)' = \frac{1}{x^2}$

$\frac{1}{x}y = -\frac{1}{x} + C$

$y = -1 + Cx$

Alternately,

$y' = \frac{1}{x} + \frac{y}{x} = \frac{1+y}{x}$

$\frac{1}{1+y}y' = \frac{1}{x}$

$\ln|1+y| = \ln|x| + c$

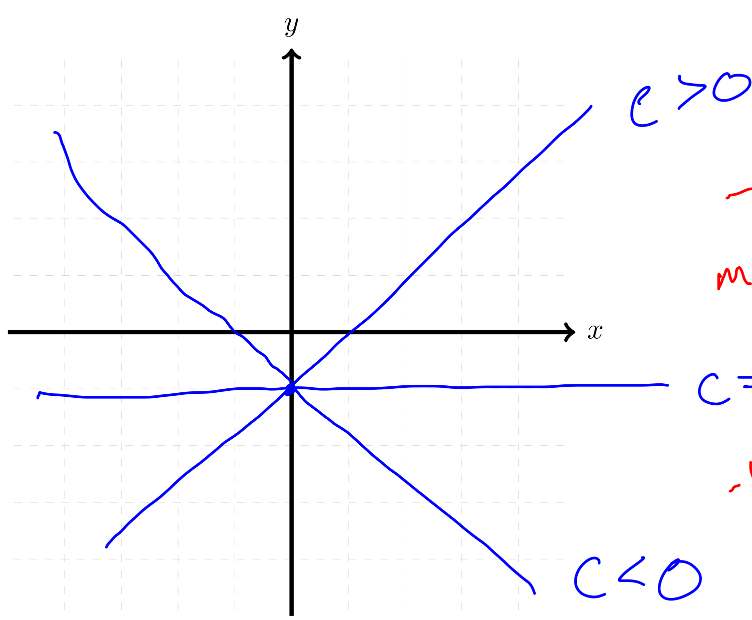
$|1+y| = |x| \cdot e^c$

$1+y = Dx$

$y = Dx - 1$

$y(x) = \boxed{Cx - 1}$

(b) **[2 pts]** Sketch integral curves of the equation (i.e. solutions) for a few characteristic values of the arbitrary constant.




-1 for missing cases

-1 or -2 for bad graphs

(c) **[1 pts]** Give an example of an initial condition that cannot be solved.

Unsolvable IC: $y\left(\boxed{0}\right) = \boxed{4}$

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