

Today

- Fourier Series examples
- The Diffusion (Heat) Equation derivation
- Solving the Diffusion equation subject to BCs and IC using FS

Deriving the FS coefficient formulae

Define the dot product for periodic functions (with period P)

$$f(x) \circ g(x) = \int_{\text{one period}} f(x) \cdot g(x) dx = \int_{-P/2}^{P/2} f(x) g(x) dx$$

Let $v_n(x) = \cos\left(\frac{2\pi n x}{P}\right)$, $w_n(x) = \sin\left(\frac{2\pi n x}{P}\right)$, $v_0(x) = 1$. ($n = 1, 2, \dots$)

Recall (or calculate for yourself) that

$$v_0(x) \circ v_0(x) = P, \quad v_m(x) \circ v_n(x) = 0 \text{ for } m \neq n, \quad v_n(x) \circ v_n(x) = P/2$$
$$w_m(x) \circ w_n(x) = 0, \quad w_m(x) \circ w_n(x) = 0 \text{ for } m \neq n, \quad w_n(x) \circ w_n(x) = P/2$$

Suppose $f(x)$ can be represented exactly as a FS. Thus

$$f(x) = A_0 v_0(x) + \sum_{m=1}^{\infty} a_m v_m(x) + \sum_{m=1}^{\infty} b_m w_m(x).$$

Find its FS coefficients. As with vectors, use 'o' to find A_0, a_n, b_n .

To find A_0 ,

$$f(x) \circ v_0(x) = A_0 v_0(x) \circ v_0(x) + \sum_{m=1}^{\infty} a_m v_m(x) \circ v_0(x) + \sum_{m=1}^{\infty} b_m w_m(x) \circ v_0(x) = A_0 \cdot P$$

$$\text{Thus, } A_0 = \frac{1}{P} f(x) \circ v_0(x) = \frac{1}{P} \int_{-P/2}^{P/2} f(x) dx.$$

To find a_n ,

$$f(x) \circ v_n(x) = A_0 v_0(x) \circ v_n(x) + \sum_{m=1}^{\infty} a_m v_m(x) \circ v_n(x) + \sum_{m=1}^{\infty} b_m w_m(x) \circ v_n(x) = a_n \underbrace{v_n(x) \circ v_n(x)}_{P/2}$$

$$\text{Thus, } a_n = \frac{2}{P} f(x) \circ v_n(x) = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \cos \frac{2n\pi x}{P} dx.$$

$$\text{Similarly, } b_n = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \sin \frac{2n\pi x}{P} dx$$

In many cases, we will have $P = 2L$ (but not always!) so

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

This pdf is also posted on the lecture slides page.

Fourier series

- Calculate the coefficients of the Fourier series of a function:

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \dots$$
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$$v_0(x) = 1$$

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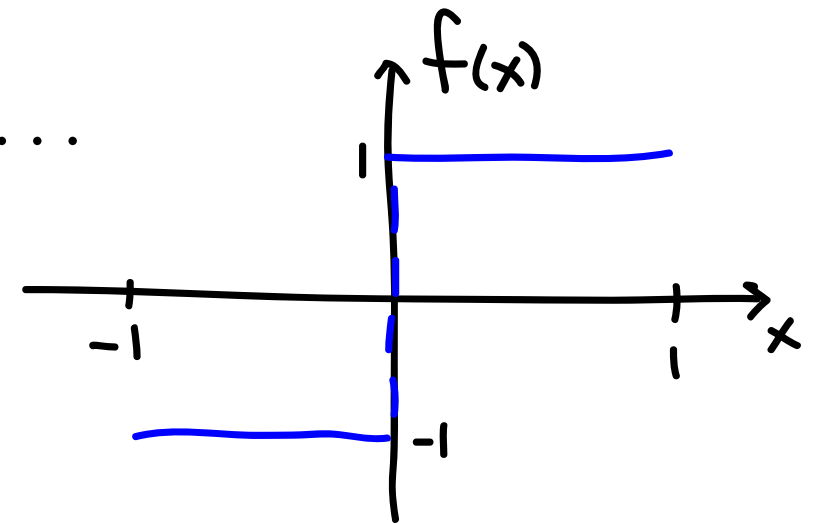
$$= a_n v_n(x) \circ v_n(x) = a_n L$$

$$a_n = \frac{1}{L} \int_{-L}^L f_{FS}(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Fourier series

- Find the Fourier series for $f(x) = 2u_0(x) - 1$ on the interval $[-1, 1]$.

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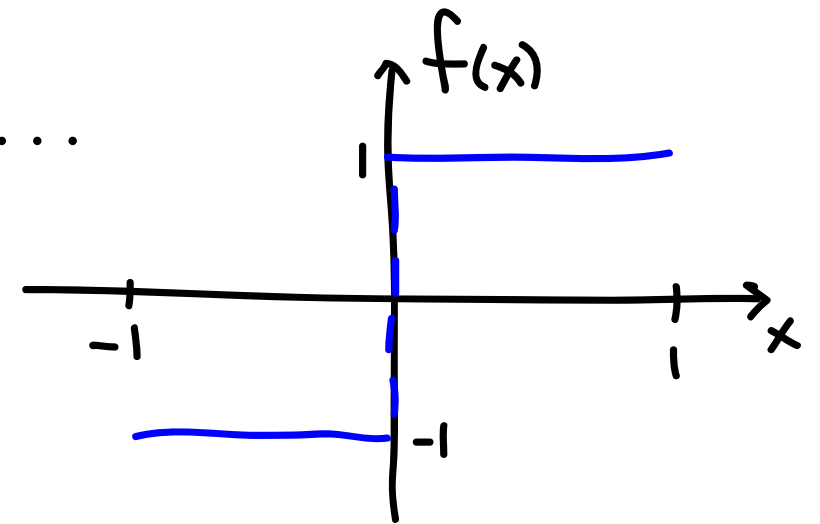


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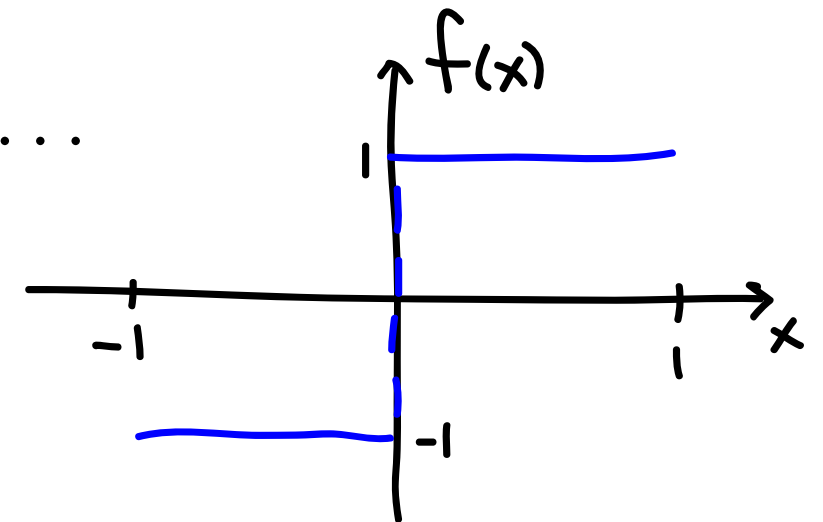
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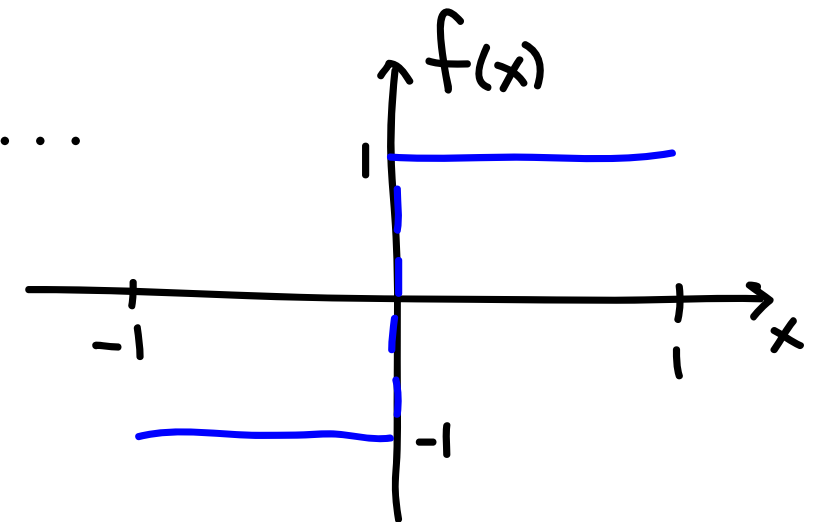
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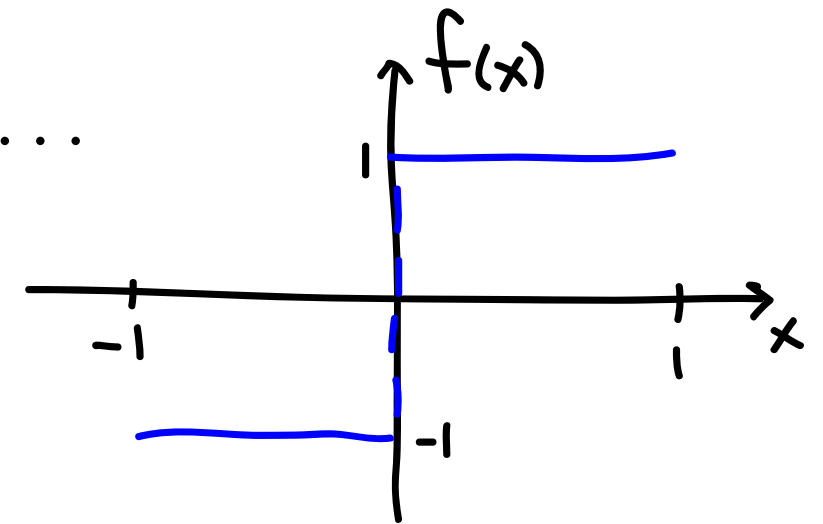
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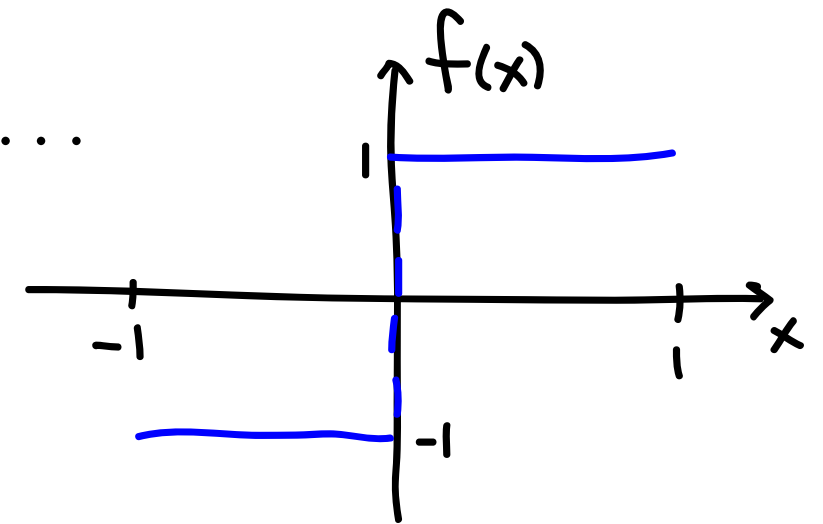
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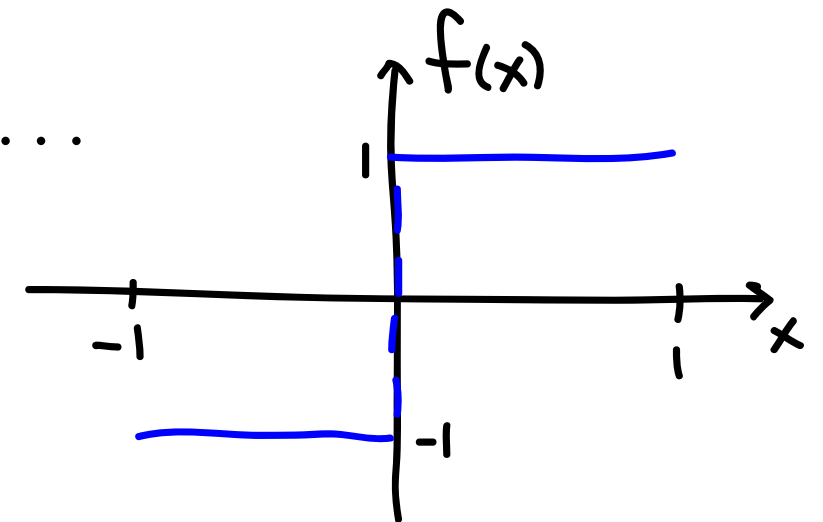
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$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx \quad \text{\textit{A}_0 is the average value of f(x)!}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

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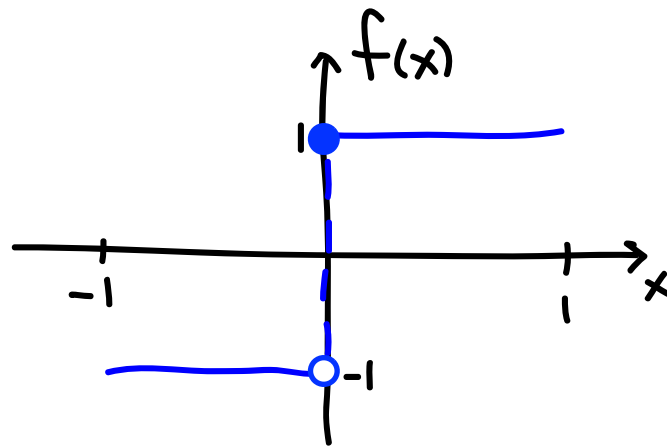
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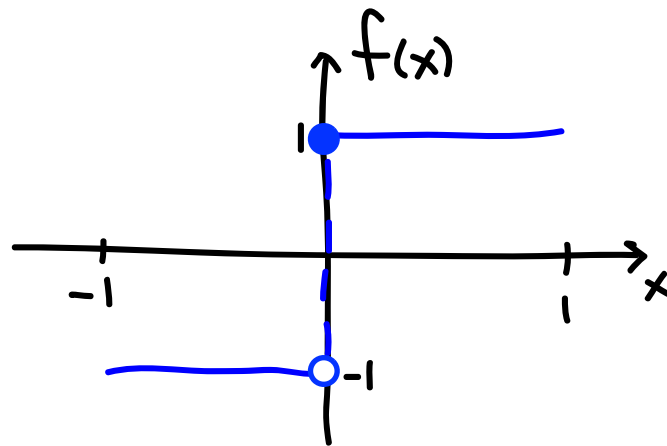


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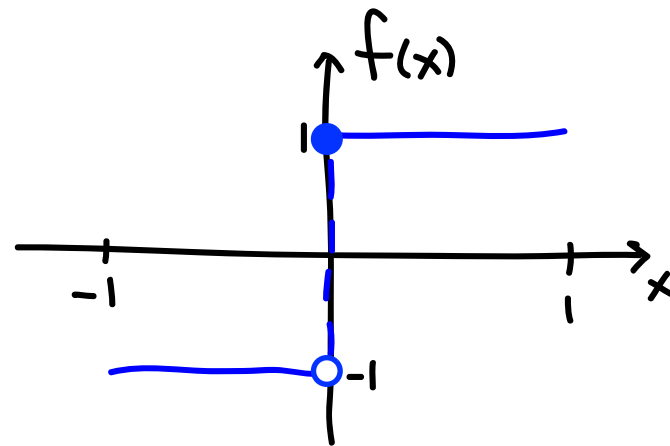
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(B) $\frac{1}{\pi}$

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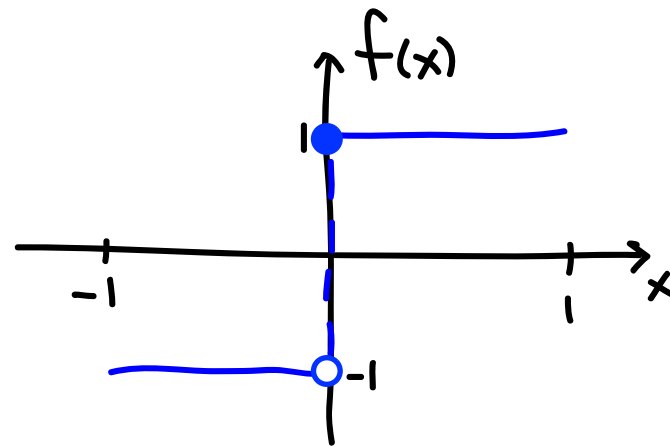
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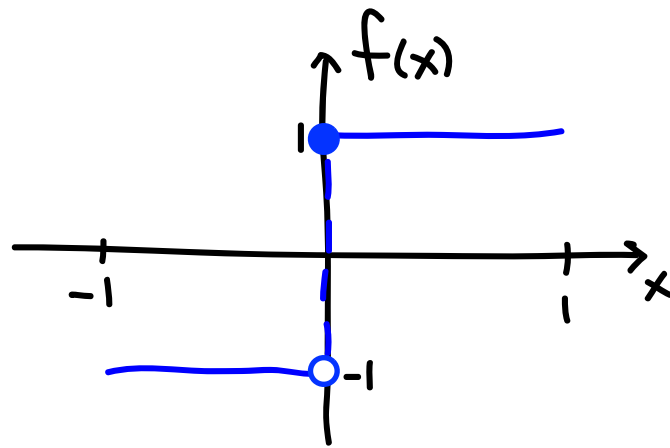


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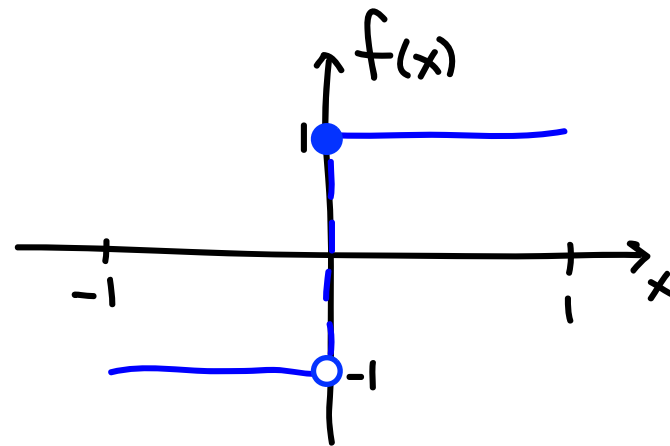
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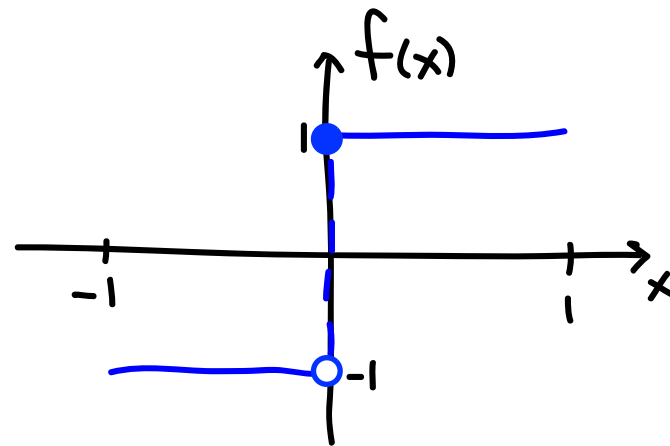
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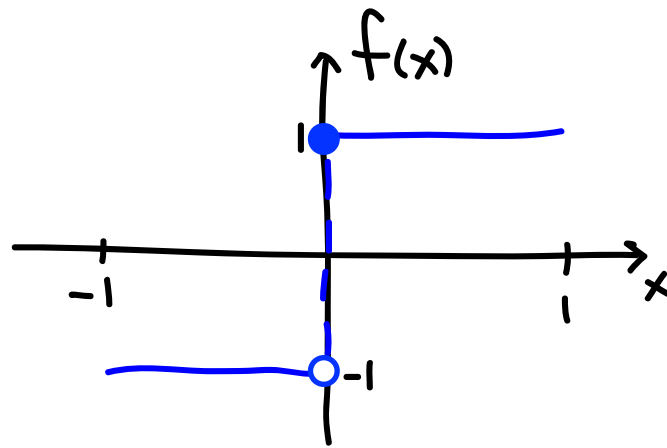


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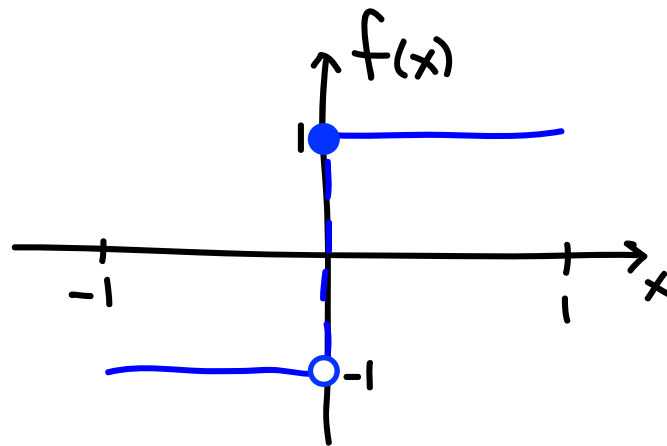
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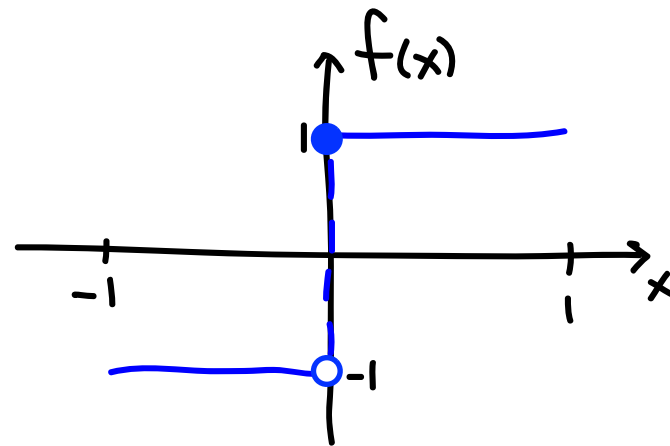
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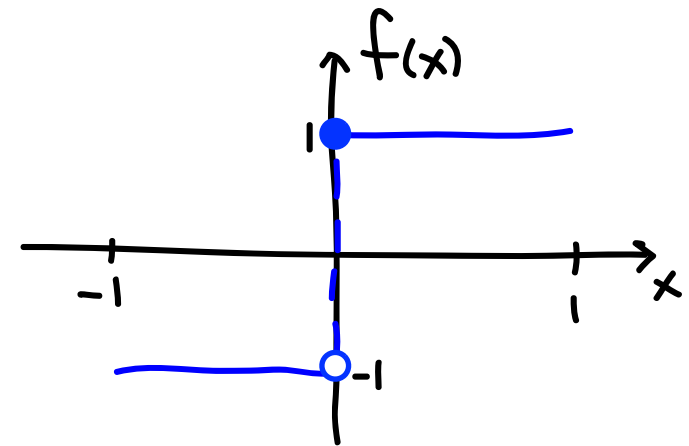
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$$b_n = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

(C) $\frac{4(-1)^n}{n\pi}$

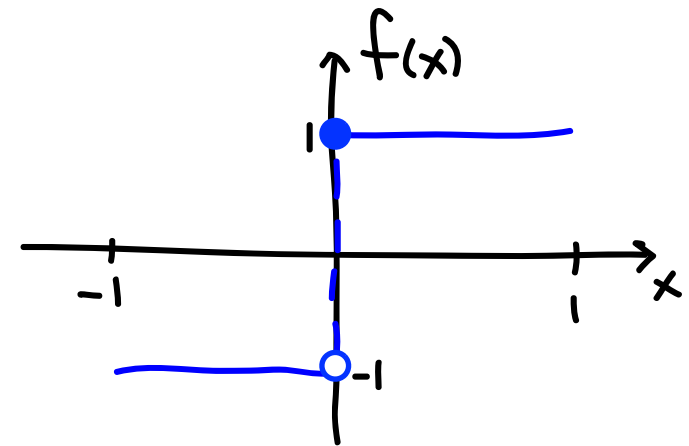
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$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \dots$$

$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \dots$$



$b_n =$

(A) 0

(B) $\frac{2}{n\pi}$

(C) $\frac{4(-1)^n}{n\pi}$

(D) $\frac{2(1 - (-1)^n)}{n\pi}$

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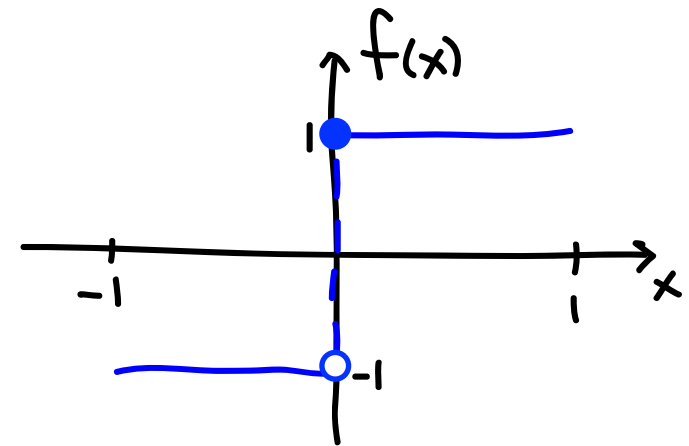
<https://www.desmos.com/calculator/tlvtikmi0y>

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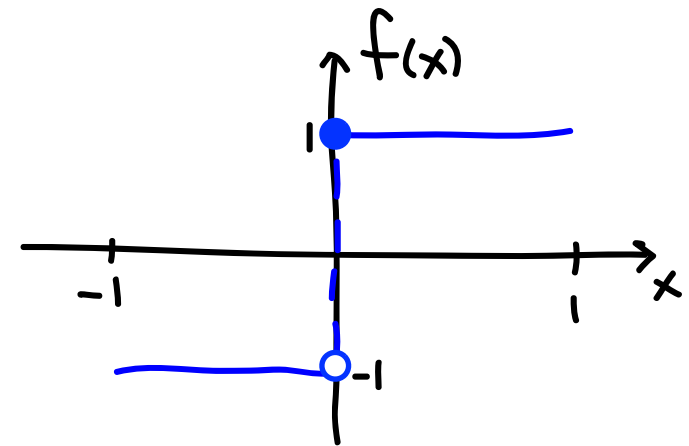
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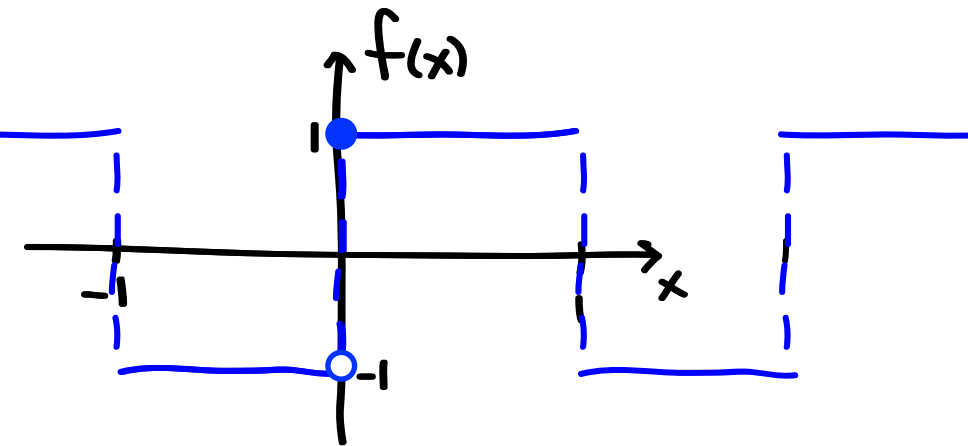
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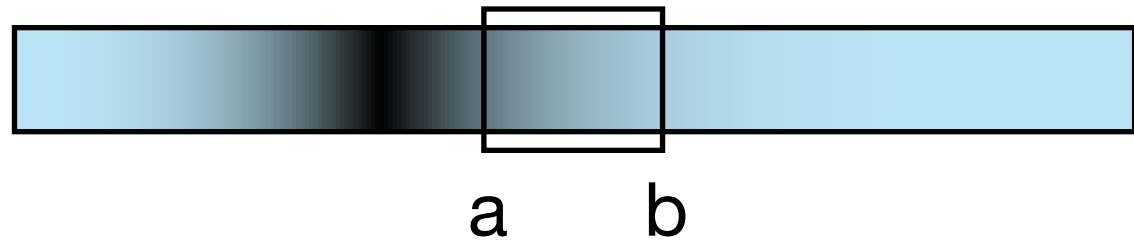
- **Theorem** Suppose f and f' are piecewise continuous on $[-L, L]$ and periodic beyond that interval. Then $f(x) = f_{FS}(x)$ at all points at which f is continuous. Furthermore, at points of discontinuity, $f_{FS}(x)$ takes the value of the midpoint of the jump. That is,

$$f_{FS}(x) = \frac{f(x^+) + f(x^-)}{2}$$

The Diffusion Equation

$c(x,t)$ is linear mass density of ink in a long narrow tube.

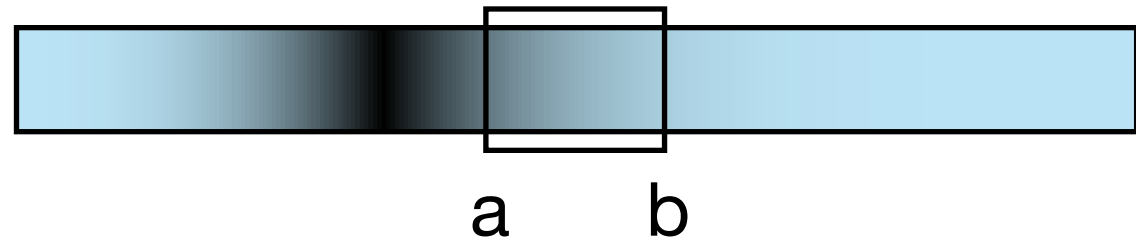
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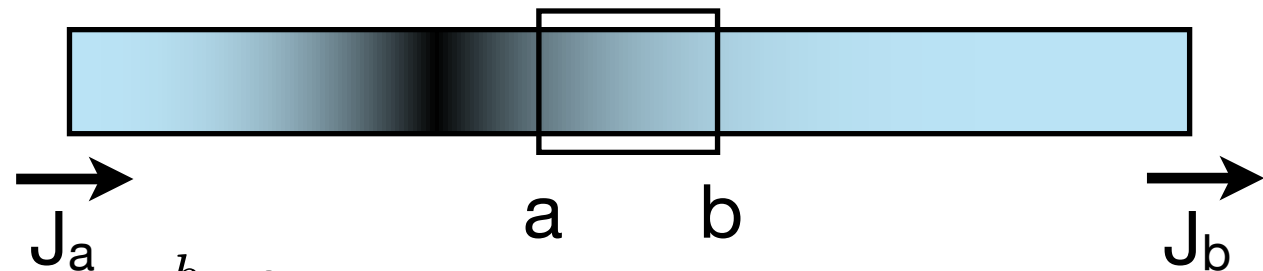
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$$\frac{dQ_{ab}}{dt}(t) = \frac{d}{dt} \int_a^b c(x,t) dx = \int_a^b \frac{\partial}{\partial t} c(x,t) dx$$

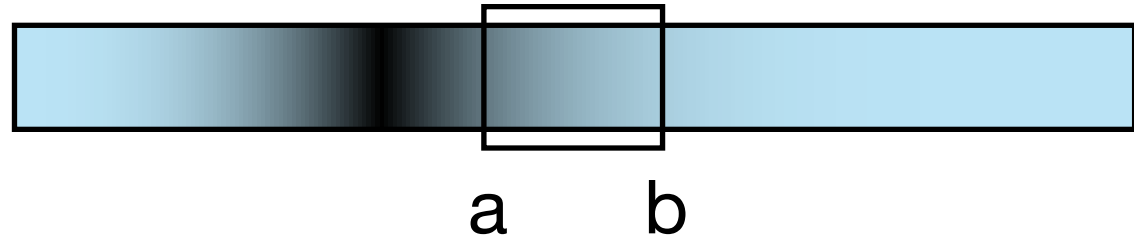
Define the flux J_a to be the amount of mass crossing the line $x=a$ per unit of time (particles moving right count as positive flux) .

In that case, the change of Q inside the a - b box can also be counted watching flux, that is, flux at a - flux at b :

$$\frac{dQ_{ab}}{dt}(t) = -J_b + J_a$$

The Diffusion Equation

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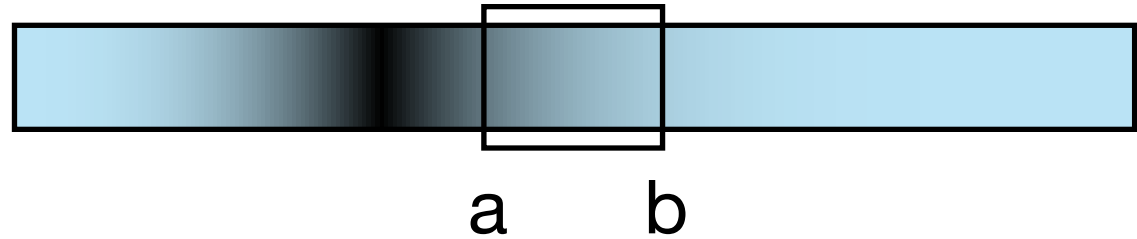
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 Need a model for flux. Let's consider simpler case first (not diffusion yet!)

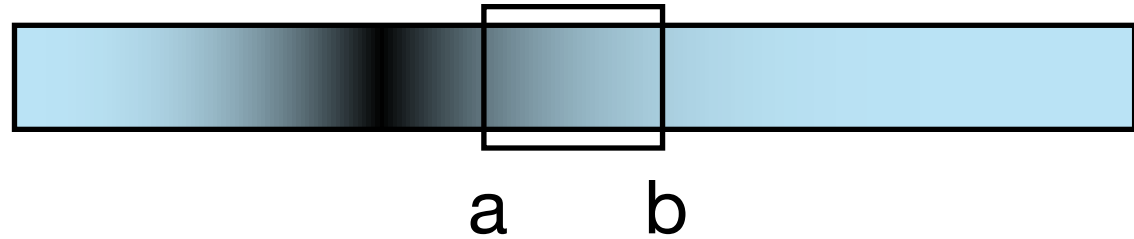
If fluid in pipe is moving with velocity v , flux is vc : $J_a = vc(a, t)$

$$\frac{dQ_{ab}}{dt}(t) = -J_b + J_a = -vc(b, t) + vc(a, t) = -vc(x, t) \Big|_a^b = - \int_a^b v \frac{\partial c}{\partial x} dx$$

$$\int_a^b \frac{\partial}{\partial t} c(x, t) dx = - \int_a^b v \frac{\partial c}{\partial x} dx \Rightarrow \frac{\partial c}{\partial t} = -v \frac{\partial c}{\partial x} \quad \text{Called Transport equation.}$$

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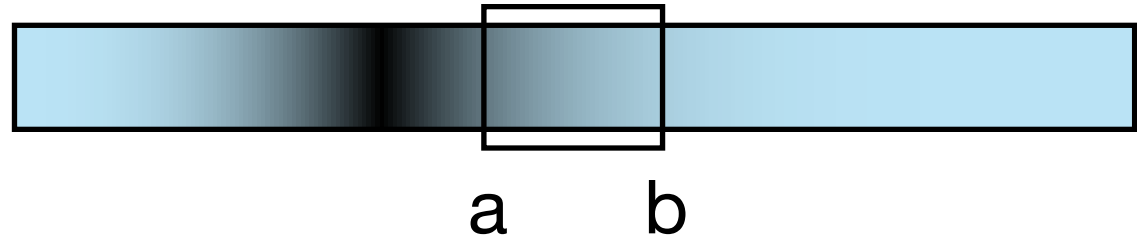
Now lets consider diffusion. We can derive this from chemical potential but

it also makes sense that for diffusion: $J_a = -D \left. \frac{\partial c}{\partial x} \right|_{x=a}$



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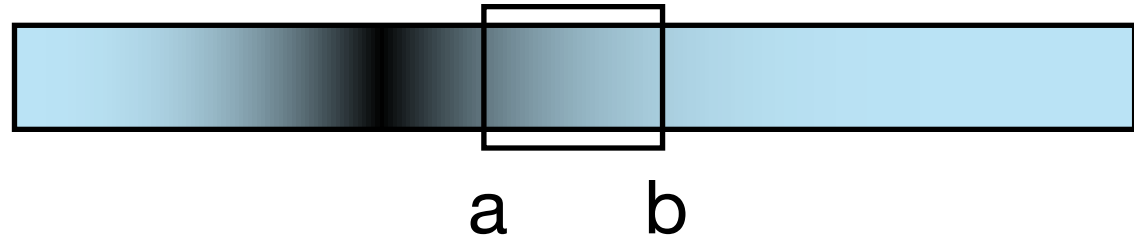
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$$\frac{dQ_{ab}}{dt}(t) = -J_b + J_a = D \left. \frac{\partial c}{\partial x} \right|_{x=b} - D \left. \frac{\partial c}{\partial x} \right|_{x=a} = D \left. \frac{\partial c}{\partial x} \right|_a^b$$

$$\int_a^b \frac{\partial}{\partial t} c(x, t) dx = \int_a^b D \frac{\partial^2 c}{\partial x^2} dx \quad \Rightarrow \quad \frac{\partial}{\partial t} c(x, t) = D \frac{\partial^2}{\partial x^2} c(x, t)$$

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The Diffusion Equation

$$\frac{dc}{dt} = D \frac{d^2 c}{dx^2}$$

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$$c_{ss}(x) = Ax + B$$

The Diffusion Equation

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The Diffusion Equation

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$$\frac{dc}{dt} = D \frac{d^2 c}{dx^2}$$

 Guess: $c(x, t) = ae^{bt} \sin(wx)$

$$\frac{\partial c}{\partial t} = abe^{bt} \sin(wx)$$

$$a = -Dw^2$$

$$D \frac{\partial^2 c}{\partial x^2} = -Daw^2 e^{bt} \sin(wx) \quad c(x, t) = ae^{-w^2 Dt} \sin(wx)$$

Still need to determine a and w . Need to impose other conditions:

- A time derivative requires an initial condition $c(x, 0)$.
- Two space derivatives require two **boundary conditions** $c(0, t)$ and $c(L, t)$.