Laplace transforms - examples

- What is the Laplace transform of $f(t) = \sin t$?

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \underset{u}{e^{-st}} \underset{dv}{\sin t \; dt}$$

$$= e^{-st}(-\cos t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st}(-\cos t)\; dt$$

$$= \lim_{A\to\infty} e^{-sA}(-\cos A) - (-1) - \int_0^\infty (-s)e^{-st}(-\cos t)\; dt$$

$$= 1 - s\int_0^\infty e^{-st}\cos t\; dt \qquad s > 0$$

$$= 1 - s\left(\underbrace{e^{-st}\sin t\Big|_0^\infty}_{0} + s\underbrace{\int_0^\infty e^{-st}\sin t\; dt}_{F(s)}\right)$$

$$= 1 - s^2F(s) \qquad s > 0$$

$$(1+s^2)F(s) = 1$$

$$F(s) = \frac{1}{1+s^2} \qquad s > 0$$

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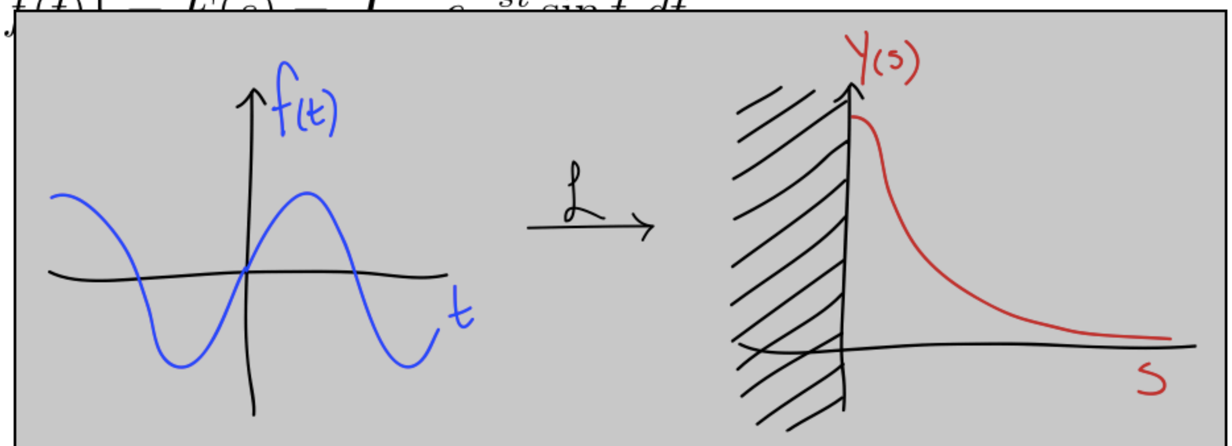

dt

$$= 1 - s \left(e^{-st} \sin t \Big|_0^\infty + s \int_0^\infty e^{-st} \sin t \; dt \right)$$

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$$(1 + s^2) F(s) = 1$$

Laplace transforms - examples

- What is the Laplace transform of $h(t) = \sin(\omega t)$? $(\omega > 0)$

$$\mathcal{L}\{h(t)\} = H(s) = \int_0^\infty e^{-st} \sin(\omega t)\ dt$$

- Hint: $u = \omega t$
  $du = \omega dt$

(A) $H(s) = \dfrac{\omega}{\omega^2 + s^2}$

(B) $H(s) = \dfrac{1}{1 + \left(\frac{s}{w}\right)^2}$

(C) $H(s) = \dfrac{1}{\omega}\ \dfrac{1}{1 + s^2}$

(D) $H(s) = \dfrac{1}{1 + s^2}$

(E) Huh?

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$$H(s) = \int_0^\infty e^{-s\frac{u}{\omega}} \sin u\ \frac{du}{\omega}$$

$$= \frac{1}{\omega} \int_0^\infty e^{-\frac{s}{\omega}u} \sin u\ du$$

$$= \frac{1}{\omega}\ F\left(\frac{s}{\omega}\right) = \frac{1}{\omega} \frac{1}{1 + \left(\frac{s}{\omega}\right)^2} \qquad s > 0$$

$$= \frac{w}{w^2 + s^2} \qquad s > 0$$

Laplace transforms - examples

- What is the Laplace transform of $g(t) = \cos t$?
- Could calculate directly but note that g(t) = f'(t) where f(t)=sin t.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\ dt$$

$$G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty \underset{u}{e^{-st}} \underset{dv}{f'(t)\ dt}$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\ dt$$

$$= -f(0) + sF(s) \qquad s > 0$$

$$= -0 + s\frac{1}{1+s^2} \quad = \frac{s}{1+s^2}$$

Laplace transforms - examples

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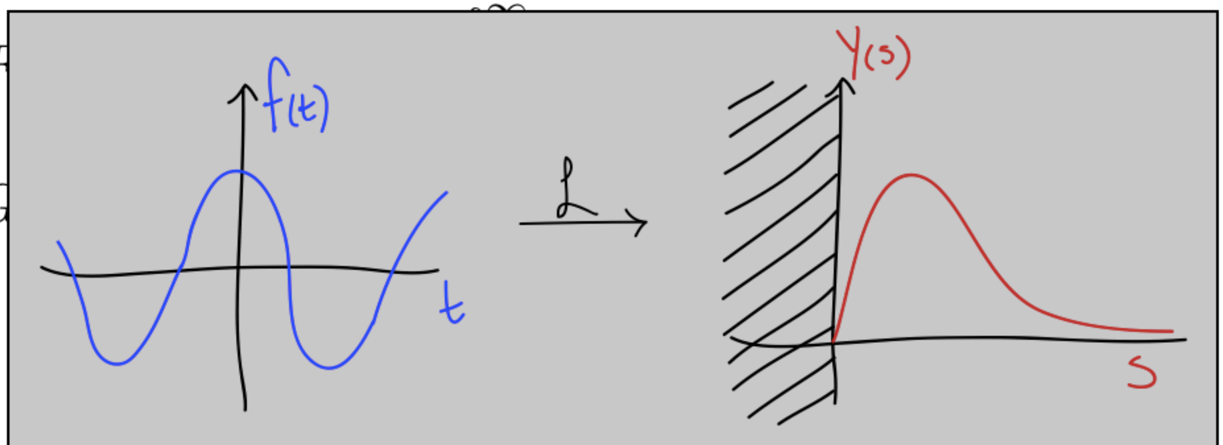

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Laplace transforms - examples

- What is the Laplace transform of $h(t) = f(\omega t)$ if $\mathcal{L}\{f(t)\} = F(s)$?

(A) $H(s) = \omega \; F(s)$

(B) $H(s) = \dfrac{1}{\omega} \; F\left(\dfrac{s}{\omega}\right)$

(C) $H(s) = \omega \; F\left(\dfrac{s}{\omega}\right)$

(D) $H(s) = \dfrac{1}{\omega} \; F(s)$

(E) Don't know.

$$\mathcal{L}\{f(\omega t)\} = \int_0^\infty e^{-st} f(\omega t) \; dt$$

- Hint: $u = \omega t$
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Laplace transforms - examples

- What is the Laplace transform of $h(t) = f(\omega t)$ if $\mathcal{L}\{f(t)\} = F(s)$?
- Recall two examples back:

$$\mathcal{L}\{h(t)\} = H(s) = \int_0^\infty e^{-st} \sin(\omega t)\ dt$$

$$u = \omega t$$
$$du = \omega dt$$

$$= \int_0^\infty e^{-s\frac{u}{\omega}} \sin u\ \frac{du}{\omega}$$

$$= \frac{1}{\omega} \int_0^\infty e^{-\frac{s}{\omega}u} \sin u\ du$$

$$= \frac{1}{\omega}\ F\left(\frac{s}{\omega}\right)$$

Laplace transforms - examples

- What is the Laplace transform of $h(t) = f(\omega t)$ if $\mathcal{L}\{f(t)\} = F(s)$?

- Recall two examples back:

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$$du = \omega dt$$

$$\begin{aligned}
\mathcal{L}\{h(t)\} = H(s) &= \int_0^\infty e^{-st}\ f(\omega t)\ \ dt \\
&= \int_0^\infty e^{-s\frac{u}{\omega}}\ f(u)\ \frac{du}{\omega} \\
&= \frac{1}{\omega}\int_0^\infty e^{-\frac{s}{\omega}u}\ f(u)\, du \\
&= \frac{1}{\omega}\ F\left(\frac{s}{\omega}\right)
\end{aligned}$$

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$$u = \omega t$$
$$du = \omega dt$$

$$= \int_0^\infty e^{-s\frac{u}{\omega}}\ f(u)\ \frac{du}{\omega}$$

$$= \frac{1}{\omega}\int_0^\infty e^{-\frac{s}{\omega}u}\ f(u)\,du$$

$$= \frac{1}{\omega}\ F\left(\frac{s}{\omega}\right)$$

$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

$$\mathcal{L}\{\cos(\omega t)\}$$
$$= \frac{1}{\omega}\ \frac{\frac{s}{\omega}}{1+\left(\frac{s}{\omega}\right)^2}$$
$$= \frac{s}{\omega^2+s^2}$$

Laplace transforms - examples

- What is the Laplace transform of $k(t) = e^{at} f(t)$ if $\mathcal{L}\{f(t)\} = F(s)$?

$$\mathcal{L}\{k(t)\} = K(s) = \int_0^\infty e^{-st} e^{at} f(t)\ dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\ dt$$

$$= F(s-a)$$

$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

$$\mathcal{L}\{e^{-3t} \cos t\} =$$

(A) $\dfrac{s}{1+(s+3)^2}$

(B) $\dfrac{1}{1+(s+3)^2}$

(C) $\dfrac{s+3}{s^2+6s+10}$

(D) $\dfrac{1}{s^2+6s+10}$

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Solving IVPs using Laplace transforms

- Solve the equation $ay'' + by' + cy = 0$ using Laplace transforms.
- Recall that $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.
- Applying this to f", we find that

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0)\\ &= s(sF(s) - f(0)) - f'(0)\\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

- Transforming both sides of the equation,

$$\mathcal{L}\{ay'' + by' + cy\} = 0$$

$$a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = 0$$

$$a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = 0$$

$$(as^2 + bs + c)Y(s) = asy(0) + ay'(0) + by(0)$$

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c}$$