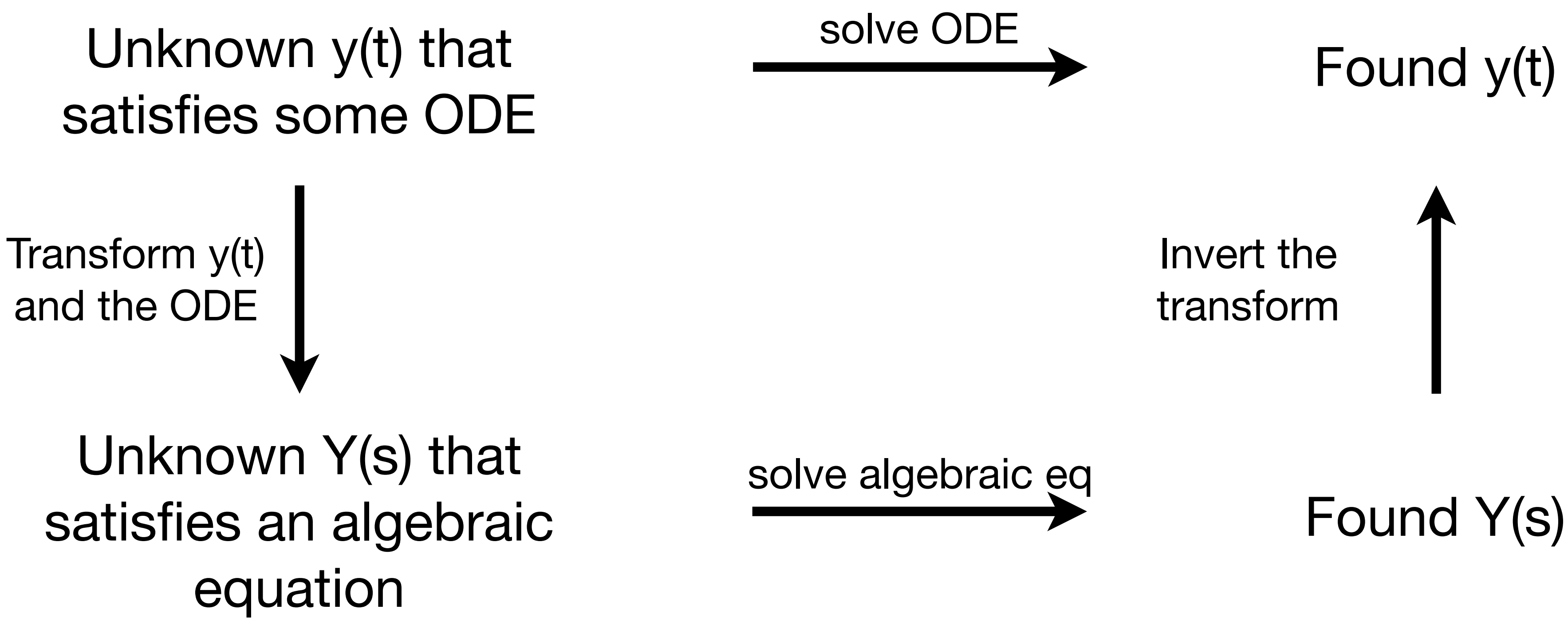
Today

- Intro to the Laplace Transform
- Solving ODEs with forcing terms using Laplace transforms - examples
- Laplace transforms of step functions
- Applications

Laplace transforms - intro (6.1)

- Using the Laplace Transform to solve (linear) ODEs.

- Idea:



- Laplace transform of y(t): $\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} y(t)\ dt$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $y(t) = 3$?

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} 3 \; dt$$

$$= -\frac{3}{s} e^{-st}\Big|_0^\infty$$

$$= \lim_{A\to\infty} -\frac{3}{s} e^{-st}\Big|_0^A$$

$$= -\frac{3}{s}\left(\lim_{A\to\infty} e^{-sA} - 1\right)$$

$$= \frac{3}{s}$$ provided $s > 0$ and does not exist otherwise.

Laplace transforms - examples (6.1)

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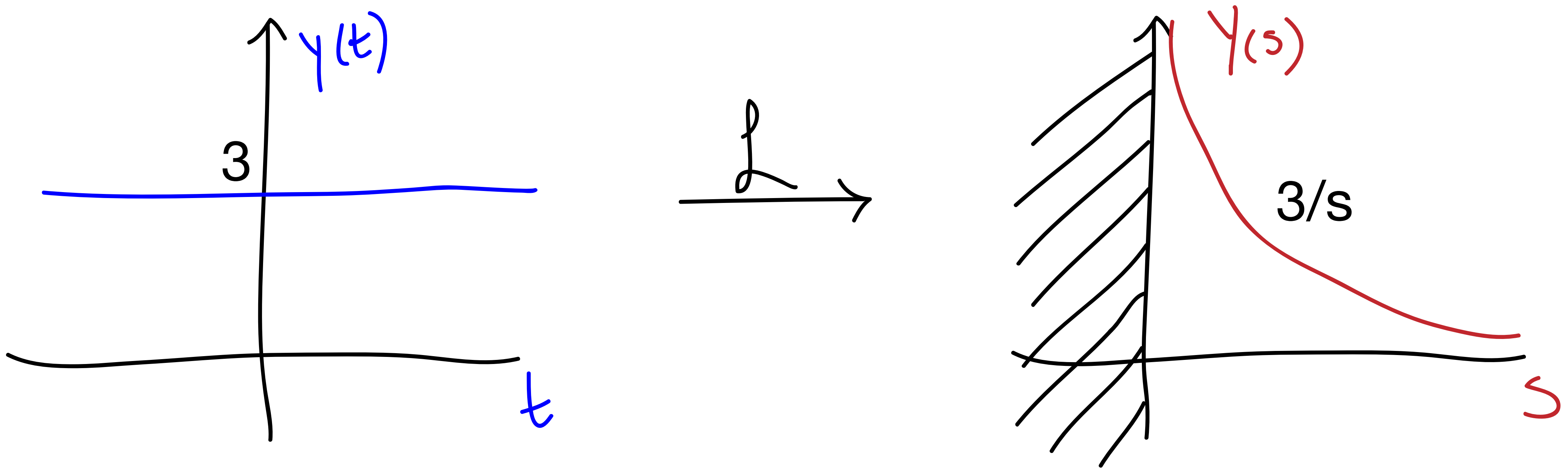
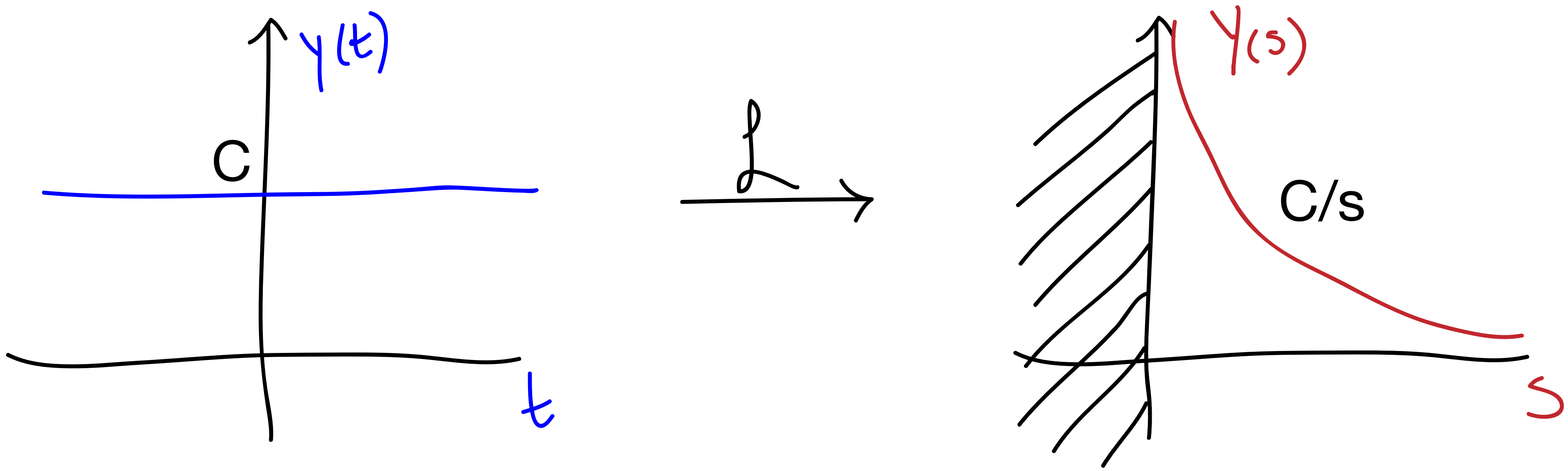


Laplace transforms - examples (6.1)

- What is the Laplace transform of $y(t) = C$?

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} C \, dt$$

$$= \frac{C}{s}$$ provided $s > 0$ and does not exist otherwise.



Laplace transforms - examples (6.1)

- What is the Laplace transform of $y(t) = e^{6t}$?

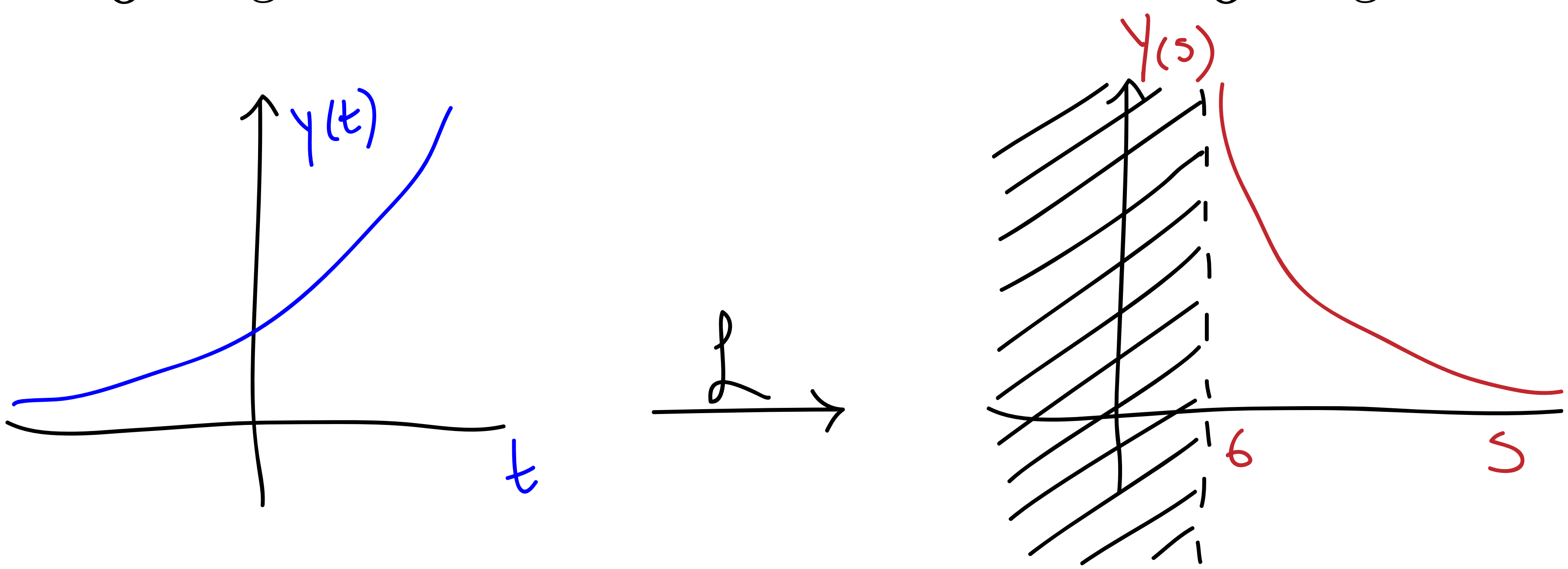
$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st} e^{6t}\, dt$$

(A) $Y(s) = \dfrac{1}{s-6} \quad s > 0$

(B) $Y(s) = \dfrac{1}{6-s} \quad s > 6$

★(C) $Y(s) = \dfrac{1}{s-6} \quad s > 6$

(D) $Y(s) = \dfrac{1}{6-s} \quad s > 0$



Laplace transforms - examples (6.1)

- What is the Laplace transform of $f(t) = \sin t$?

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \underset{u}{e^{-st}} \underset{dv}{\sin t \; dt}$$

$$= e^{-st}(-\cos t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st}(-\cos t)\; dt$$

$$= \lim_{A\to\infty} e^{-sA}(-\cos A) - (-1) - \int_0^\infty (-s)e^{-st}(-\cos t)\; dt$$

$$= 1 - s\int_0^\infty e^{-st}\cos t\; dt \qquad s > 0$$

$$= 1 - s\left(\underbrace{e^{-st}\sin t\Big|_0^\infty}_{0} + s\underbrace{\int_0^\infty e^{-st}\sin t\; dt}_{F(s)} \right)$$

$$= 1 - s^2 F(s) \qquad s > 0$$

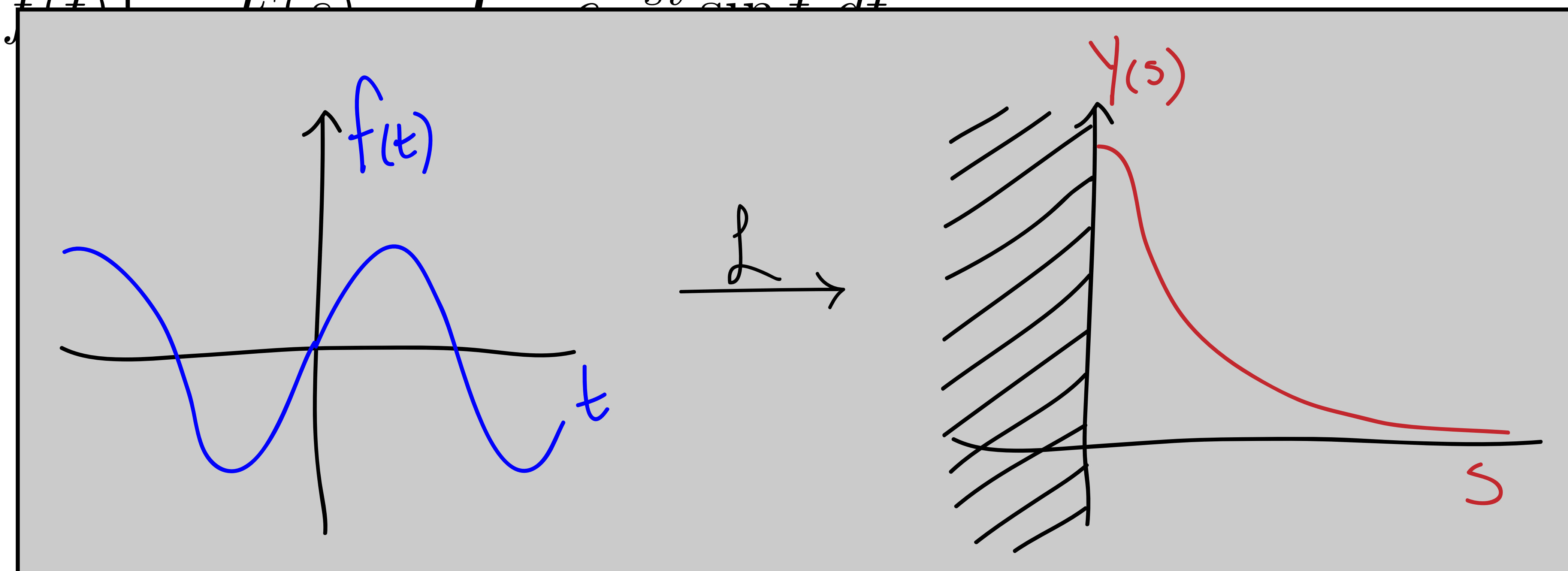
$$(1+s^2)F(s) = 1$$

$$F(s) = \frac{1}{1+s^2} \qquad s > 0$$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $f(t) = \sin t$?

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} \sin t \; dt$$



$$dt$$

$$= 1 - s \left(e^{-st} \sin t \Big|_0^\infty + s \int_0^\infty e^{-st} \sin t \; dt \right)$$

$$= 1 - s^2 F(s) \qquad s > 0$$

$$(1 + s^2) F(s) = 1$$

$$F(s) = \frac{1}{1 + s^2} \qquad s > 0$$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $h(t) = \sin(\omega t)$? $\quad (\omega > 0)$

$$\mathcal{L}\{h(t)\} = H(s) = \int_0^\infty e^{-st}\sin(\omega t)\ dt$$

$$u = \omega t$$
$$du = \omega dt$$

★ (A) $H(s) = \dfrac{\omega}{\omega^2 + s^2}$

(B) $H(s) = \dfrac{1}{1 + \left(\frac{s}{w}\right)^2}$

(C) $H(s) = \dfrac{1}{\omega}\ \dfrac{1}{1 + s^2}$

(D) $H(s) = \dfrac{1}{1 + s^2}$

(E) Huh?

$$H(s) = \int_0^\infty e^{-s\frac{u}{\omega}}\sin u\ \frac{du}{\omega}$$

$$= \frac{1}{\omega}\int_0^\infty e^{-\frac{s}{\omega}u}\sin u\ du$$

$$= \frac{1}{\omega}\ F\left(\frac{s}{\omega}\right)$$

$$= \frac{1}{\omega}\ \frac{1}{1 + \left(\frac{s}{\omega}\right)^2} \qquad s > 0$$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $g(t) = \cos t$?
- Could calculate directly but note that g(t) = f'(t) where f(t)=sin t.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\ dt$$

$$G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty \underset{u}{e^{-st}} \underset{dv}{f'(t)\ dt}$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\ dt$$

$$= -f(0) + sF(s) \qquad s > 0$$

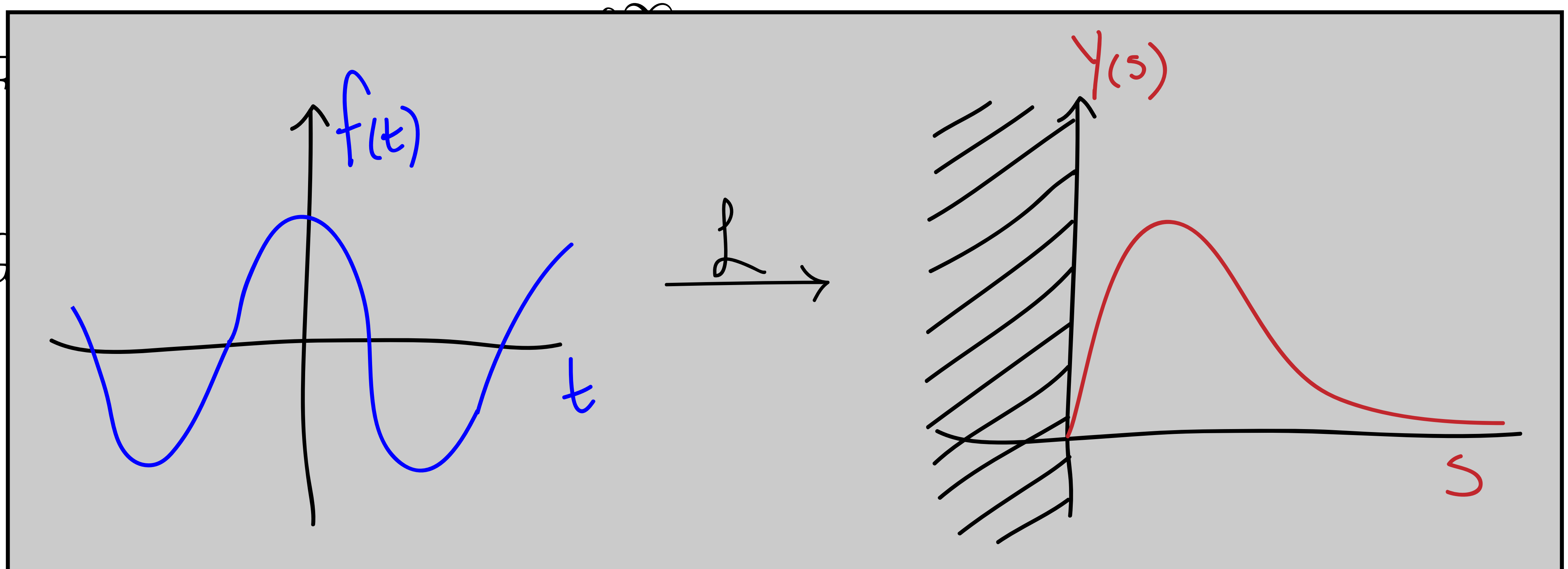
$$= -0 + s\frac{1}{1+s^2} = \frac{s}{1+s^2}$$

Laplace transforms - examples (6.1)

- What is the Lapla

$$G(s) = \mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

- Could calculate directly but note that g(t) = f'(t) where f(t)=sin t.



$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$s > 0$

$1 + s^2 \qquad 1 + s^2$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $h(t) = f(\omega t)$ if $\mathcal{L}\{f(t)\} = F(s)$?

- Recall two examples back:

$$u = \omega t$$
$$du = \omega dt$$

$$\mathcal{L}\{h(t)\} = H(s) = \int_0^\infty e^{-st}\ f(\omega t)\ \ dt$$
$$= \int_0^\infty e^{-s\frac{u}{\omega}}\ f(u)\ \frac{du}{\omega}$$
$$= \frac{1}{\omega}\int_0^\infty e^{-\frac{s}{\omega}u}\ f(u)\,du$$
$$= \frac{1}{\omega}\ F\left(\frac{s}{\omega}\right)$$

$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

$$\mathcal{L}\{\cos(\omega t)\}$$
$$= \frac{1}{\omega}\ \frac{\frac{s}{\omega}}{1+\left(\frac{s}{\omega}\right)^2}$$
$$= \frac{s}{\omega^2+s^2}$$

Laplace transforms - examples (6.1)

- What is the Laplace transform of $k(t) = e^{at} f(t)$ if $\mathcal{L}\{f(t)\} = F(s)$?

$$\mathcal{L}\{k(t)\} = K(s) = \int_0^\infty e^{-st} e^{at} f(t)\ dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\ dt$$

$$= F(s-a)$$

$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

$$\mathcal{L}\{e^{-3t} \cos t\} =$$

(A) $\dfrac{s}{1+(s+3)^2}$

(B) $\dfrac{1}{1+(s+3)^2}$

☆(C) $\dfrac{s+3}{s^2+6s+10}$

(D) $\dfrac{1}{s^2+6s+10}$

Solving IVPs using Laplace transforms (6.2)

- Solve the equation $ay'' + by' + cy = 0$ using Laplace transforms.
- Recall that $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.
- Applying this to f”, we find that

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0) \\ &= s(sF(s) - f(0)) - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

- Transforming both sides of the equation,

$$\mathcal{L}\{ay'' + by' + cy\} = 0$$

$$a\underline{\mathcal{L}\{y''\}} + b\underline{\mathcal{L}\{y'\}} + c\underline{\mathcal{L}\{y\}} = 0$$

$$a(\underline{s^2Y(s) - sy(0) - y'(0)}) + b(\underline{sY(s) - y(0)}) + c\underline{Y(s)} = 0$$

$$(as^2 + bs + c)Y(s) = asy(0) + ay'(0) + by(0)$$

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c}$$

Solving IVPs using Laplace transforms (6.2)

- Solve the equation $y'' + 4y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c}$$

$$= \frac{s}{s^2+4}$$

$$a = 1$$
$$b = 0$$
$$c = 4$$

- To find y(t), we have to invert the transform. What y(t) would have Y(s) as its transform?

- Recall that $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{\omega^2 + s^2}$. So $y(t) = \cos(2t)$.

Solving IVPs using Laplace transforms (6.2)

- Solve the equation $y'' + 6y' + 13y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 \ldots} \longrightarrow Y(s) = \frac{s+6}{s^2 + 6s + 13}$$

$$\lambda = \frac{-6 \pm i\sqrt{52-36}}{2} = -3 \pm 2i$$

- To find y(t), we hav… would have Y(s) as its transform?

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}\{e^{-3t}\cos t\} = \frac{s+3}{1+(s+3)^2}$$

$$Y(s) = \frac{s+3+3}{s^2+6s+9+4}$$

$$= \frac{s+3}{(s+3)^2+4} + \frac{3}{(s+3)^2+4}$$

$$= \frac{s+3}{(s+3)^2+4} + \frac{3}{2}\frac{2}{(s+3)^2+4}$$

$$y(t) = e^{-3t}\cos(2t) + \frac{3}{2}e^{-3t}\sin(2t)$$

Solving IVPs using Laplace transforms (6.2)

- Solve the equation $y'' + 6y' + 13y = 0$ with initial conditions y(0)=1, y'(0)=0 using Laplace transforms.

$$Y(s) = \frac{s+6}{s^2+6s+13} = \frac{s+6}{s^2+6s+9+4} = \frac{s+6}{(s+3)^2+4}$$

$$= \frac{s+3+3}{(s+3)^2+4} = \frac{s+3}{(s+3)^2+4} + \frac{3}{(s+3)^2+4}$$

$$= \frac{s+3}{(s+3)^2+2^2} + \frac{3}{2}\frac{2}{(s+3)^2+2^2}$$

1. Does the denominator have real or complex roots? Complex.

2. Complete the square.

3. Put numerator in form (s+α)+β where (s+α) is the completed square.

4. Fix up coefficient of the term with no s in the numerator. 5. Invert.