Today

- Office hour Jan 17 cancelled - Jan 18 instead, 12-1pm
- Saltwater inflow example
- General solutions, independence of functions and the Wronskian
- Distinct roots of the characteristic equation
- Review of complex numbers
- Complex roots of the characteristic equation

Modeling - Example

- Saltwater with a concentration of 200 g/L flows into a tank at a rate 2 L/min. The tank starts with no salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

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(A) $m' = 200 - 2m, \quad m(0) = 0$

(B) $m' = 400 - 2m, \quad m(0) = 200$

(C) $m' = 400 - m/5, \quad m(0) = 0$

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- What happens when $m < 2000$? ---> $m' > 0$.
- What happens when $m > 2000$? ---> $m' < 0$.
- Limiting mass: 2000 g (Long way: solve the eq. and let $t \to \infty$.)

Existence and uniqueness

Theorem 2.4.2 Let the functions f and $\dfrac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta, \quad \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t_0 < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the IVP

$$y' = f(t, y), \quad y(t_0) = y_0.$$

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- A couple questions/examples to explore on your own:
  - Why don't we get a solution all the way to the ends of the t interval?
    - Example: $$\frac{dy}{dt} = y^2, \quad y(0) = 1$$
  - How does a non-continuous RHS lead to more than one solution?
    - Example: $$\frac{dy}{dt} = \sqrt{y}, \quad y(0) = 0$$

Second order linear equations

- The general form for a second order linear equation:

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- As with first order linear equations, we have homogeneous (g=0) and non-homogeneous second order linear equations.

- We'll start by considering the homogeneous case with constant coefficients:

$$ay'' + by' + cy = 0$$

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  (A) $y(t) = y_1(t)^2$

  (B) $y(t) = y_1(t)+y_2(t)$

  (C) $y(t) = y_1(t)\ y_2(t)$

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- The general solution will be $y(t) = C_1y_1(t) + C_2y_2(t)$.

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$$e^t y' + e^t y = C_1 e^t$$

$$(e^t y)' = C_1 e^t$$

$$e^t y = C_1 e^t + C_2$$

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- More common would be that we find solutions y(t) = 1 and y(t)= e^{-t} and simply write down

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$$(e^{rt})'' + (e^{rt})' = 0$$
$$r^2 e^{rt} + r e^{rt} = 0$$
$$r^2 + r = 0$$
$$r(r+1) = 0$$
$$r = 0, \quad r = -1$$

$$y = C_1 e^0 + C_2 e^{-t}$$
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$$(e^{rt})'' - 4(e^{rt}) = 0$$

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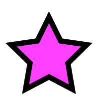 (B) $y(t) = 2e^{2t} + e^{-2t}$

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$$y(0) = C_1 + C_2 = 3$$

$$y'(0) = 2C_1 - 2C_2 = 2$$

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    - Two distinct real roots: $b^2 - 4ac > 0$. ($r_1 \neq r_2$)
    - A repeated real root: $b^2 - 4ac = 0$.
    - Two complex roots: $b^2 - 4ac < 0$.

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- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

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    - Two distinct real roots: $b^2 - 4ac > 0$. ($r_1 \neq r_2$)
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- For case i, we get $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.
- Do our two solutions cover all possible ICs? That is, can we use them to form a general solution?

Independence and the Wronskian (Section 3.2)

- Example: Suppose $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are two solutions to some equation. Can we solve ANY initial condition $y(0) = y_0, \ y'(0) = v_0$ with these two solutions?

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$$\det \begin{pmatrix} e^3 & e^{-3} \\ 2e^3 & 2e^{-3} \end{pmatrix} = 0$$

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- For any two solutions to some linear ODE, to ensure that we have a general solution, we need to check that

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- This quantity is called the Wronskian.

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  e.g. $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are not independent.

  Find values of $C_1 \neq 0$ and $C_2 \neq 0$ so that $C_1y_1(t) + C_2y_2(t) = 0$.

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(C) $C_1 = e^{-3}, \ C_2 = e^{3}$

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- If $y_1(t)$ and $y_2(t)$ are solutions to an ODE and the Wronskian is nonzero then they are independent and

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

is the general solution. We call $y_1(t)$ and $y_2(t)$ a fundamental set of solutions and we can use them to solve any IC.

Independence and the Wronskian (Section 3.2)

- So for case i (distinct roots), can we form a general solution from

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$$= (r_1 - r_2) e^{r_1 t} e^{r_2 t} \neq 0$$

So yes! $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is the general solution.