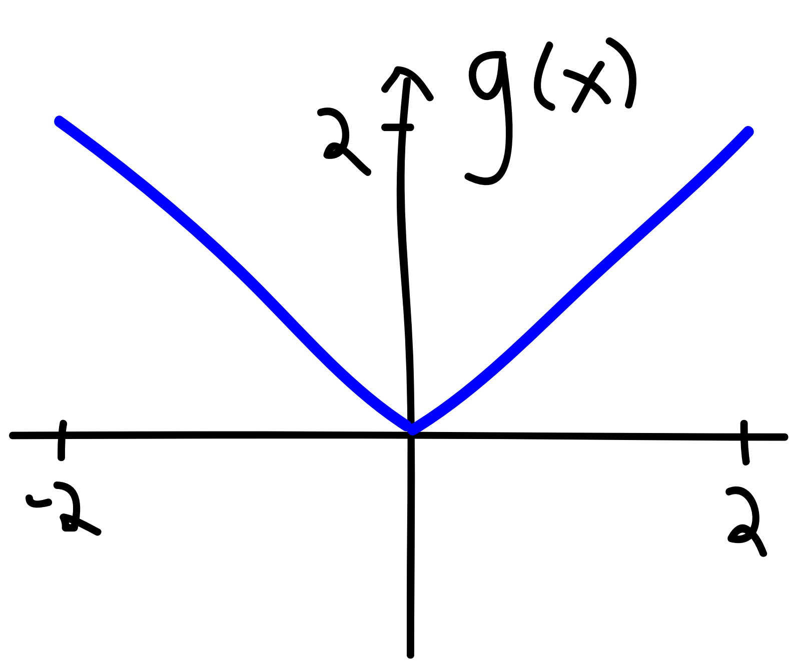
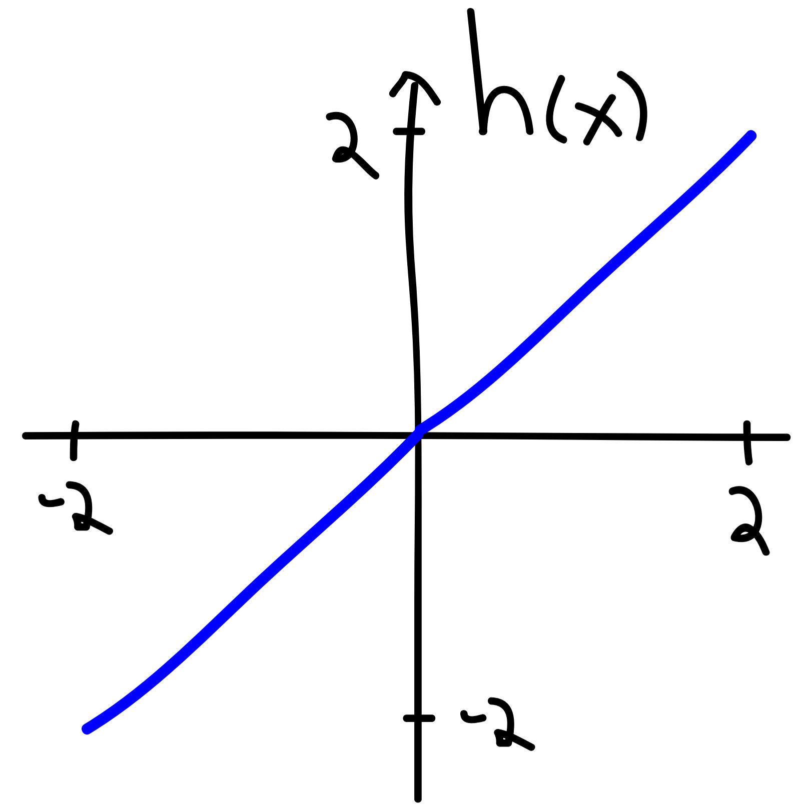
Today

- Fourier Series examples - even and odd extensions, other symmetries
- Using Fourier Series to solve the Diffusion Equation

Examples - calculate the Fourier Series





$$a_n = \begin{cases} 0 & n \text{ even} \\ -\dfrac{8}{n^2\pi^2} & n \text{ odd} \end{cases}$$

$$g(x) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$= \sum_{k=1}^{\infty} a_{2k-1} \cos \frac{(2k-1)\pi x}{L}$$

$$= 1 - \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos \frac{(2k-1)\pi x}{2}$$

$$b_n = \frac{(-1)^{n+1}\,4}{n\pi}$$

$$h(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2}$$

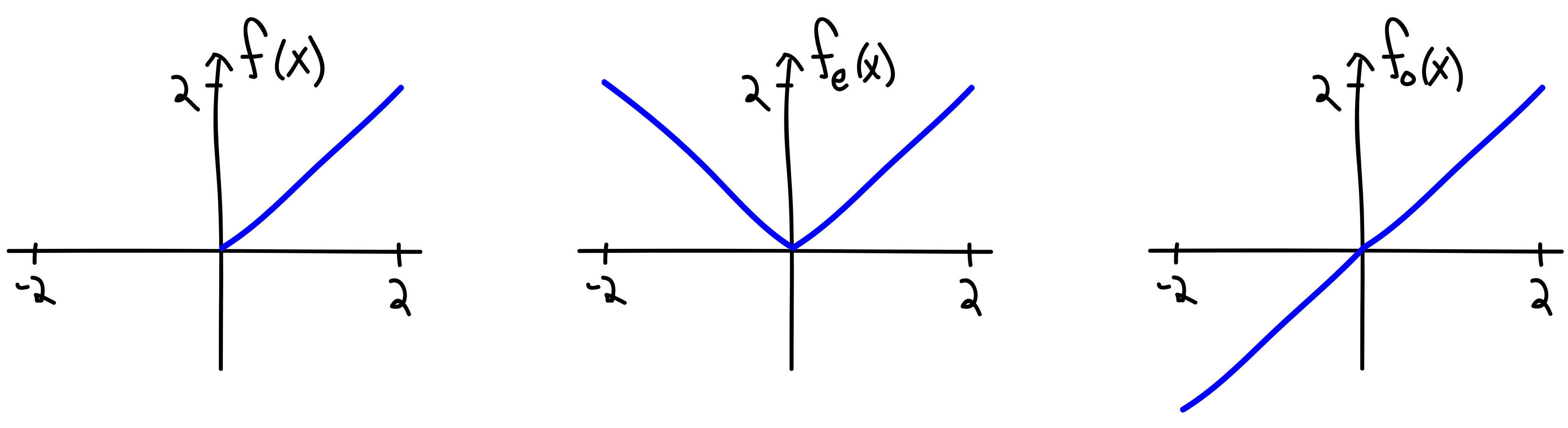
Even and odd extensions

- For a function *f(x)* defined on [0,L], the even extension of *f(x)* is the function

$$f_e(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq L, \\ f(-x) & \text{for } -L \leq x < 0. \end{cases}$$

- For a function *f(x)* defined on [0,L], the odd extension of *f(x)* is the function

$$f_o(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq L, \\ -f(-x) & \text{for } -L \leq x < 0. \end{cases}$$



Even and odd extensions

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- Because these functions are even/odd, their Fourier Series have a couple simplifying features:

no sin

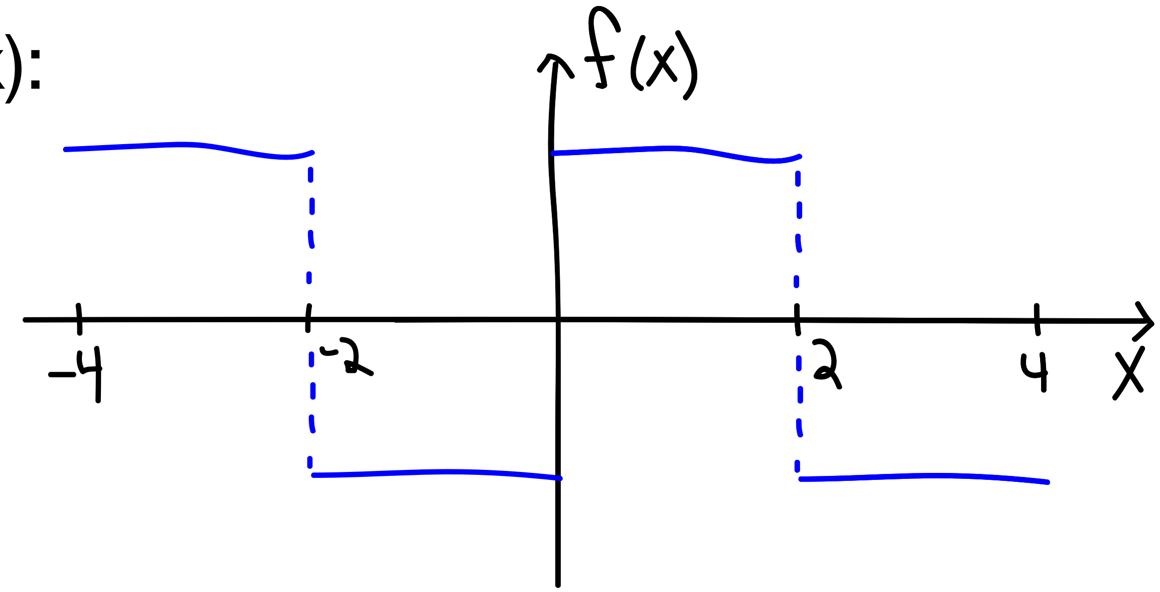
$$f_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

no cos

$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Fourier Series for functions with other symmetries

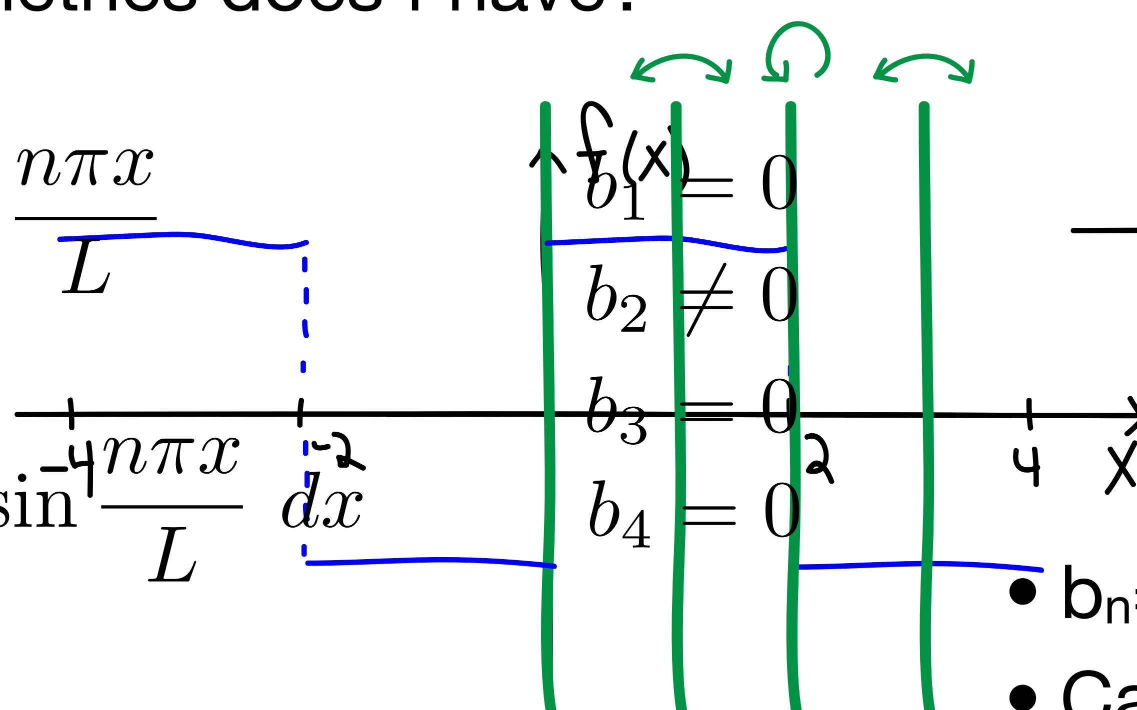
- Find the Fourier Sine Series for f(x):



- Because we want the sine series, we use the odd extension.
- The Fourier Series for the odd extension has a_n=0 because of the symmetry about x=0.
- What other symmetries does f have?

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

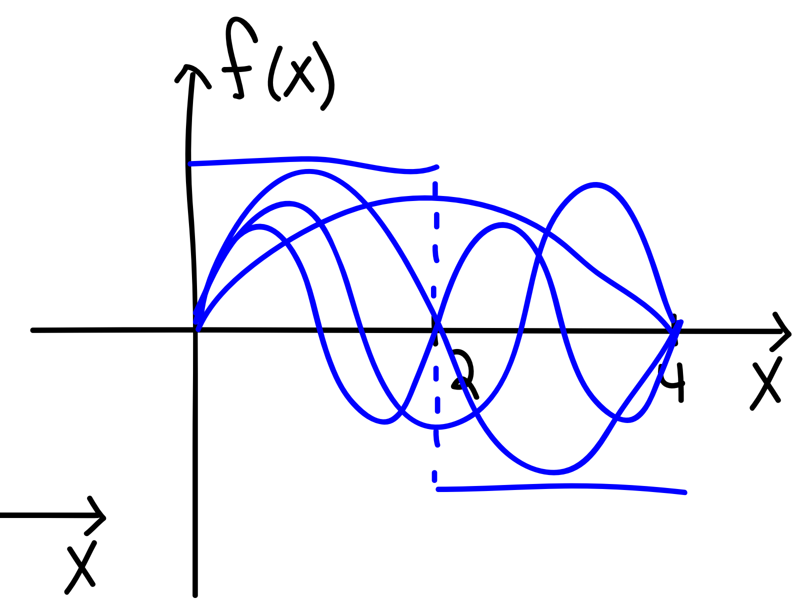


$b_1 = 0$

$b_2 \neq 0$

$b_3 = 0$

$b_4 = 0$



- b_n=0 for n = odd or 4k
- Calculate b_n

Using Fourier Series to solve the Diffusion Equation

$$u_t = 4u_{xx}$$

$$\left.\frac{du}{dx}\right|_{x=0,2} = 0$$

$$u(x,0) = \cos\frac{3\pi x}{2}$$

The IC is an eigenvector! Note that it satisfies the BCs.

$$v_3(x) = \cos\frac{3\pi x}{2}$$

$$v_n(x) = \cos\frac{n\pi x}{2}$$

$$u_n(x,t) = e^{\lambda_n t}\cos\frac{n\pi x}{2}$$

$$\frac{\partial}{\partial t}u_n(x,t) = \lambda_n e^{\lambda_n t}\cos\frac{n\pi x}{2}$$

$$4\frac{\partial^2}{\partial x^2}u_n(x,t) = -\frac{4n^2\pi^2}{4}e^{\lambda_n t}\cos\frac{n\pi x}{2}$$

$$\lambda_n = -n^2\pi^2$$

So the solution is

$$u(x,t) = e^{-9\pi^2 t}\cos\frac{3\pi x}{2}$$

Using Fourier Series to solve the Diffusion Equation

$$u_t = 4u_{xx}$$

$$\left.\frac{du}{dx}\right|_{x=0,2} = 0$$

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

The IC is the sum of eigenvectors!

$$u_n(x,t) = e^{\lambda_n t}\cos\frac{n\pi x}{2}$$

$$\lambda_n = -n^2\pi^2$$

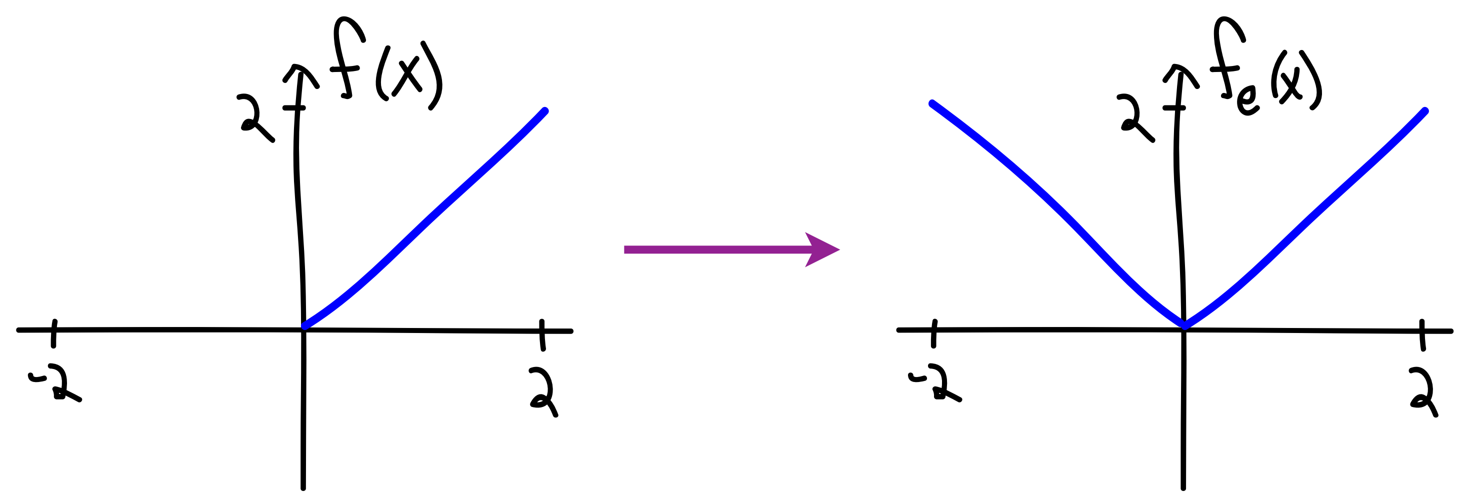
$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\cos\frac{n\pi x}{2}$$

Using Fourier Series to solve the Diffusion Equation

$$u_t = 4u_{xx}$$

$$\left.\frac{du}{dx}\right|_{x=0,2} = 0$$

$$u(x,0) = x$$



★ (A) $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos\frac{n\pi x}{2}$ $a_0 = 1,\ a_n = -\frac{8}{n^2\pi^2}$ for n even (0 for n odd)

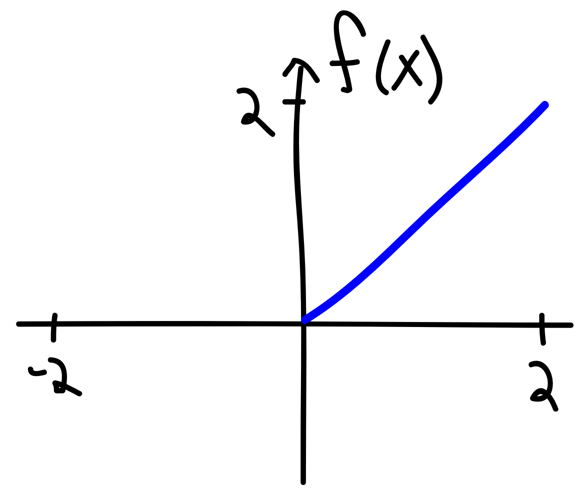
(B) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin\frac{n\pi x}{2}$ $b_n = \frac{(-1)^{n+1}4}{n\pi}$

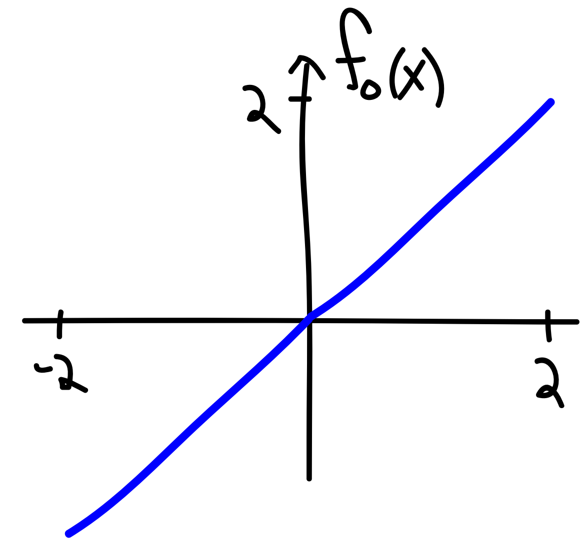
Using Fourier Series to solve the Diffusion Equation

$$u_t = 4u_{xx}$$

$$u(0,t) = u(2,t) = 0$$

$$u(x,0) = x$$





(A) $$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos\frac{n\pi x}{2}$$

$$a_0 = 1,\ a_n = -\frac{8}{n^2\pi^2} \text{ for } n \text{ even}$$
$$(0 \text{ for } n \text{ odd})$$

★ (B) $$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin\frac{n\pi x}{2}$$

$$b_n = \frac{(-1)^{n+1}4}{n\pi}$$

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