Today

- The geometry of homogeneous and nonhomogeneous matrix equations
- Solving nonhomogeneous equations
  - Method of undetermined coefficients

Second order, linear, constant coeff, **non**homogeneous (3.5)

- Our next goal is to figure out how to find solutions to nonhomogeneous equations like this one:

$$y'' - 6y' + 8y = \sin(2t)$$

- But first, a bit more on the connections between matrix algebra and differential equations . . .

Some connections to linear (matrix) algebra

- An mxn matrix is a gizmo that takes an n-vector and returns an m-vector:

$$\overline{y} = A\overline{x}$$

- It is called a linear operator because it has the following properties:

$$A(c\overline{x}) = cA\overline{x}$$

$$A(\overline{x} + \overline{y}) = A\overline{x} + A\overline{y}$$

- Not all operators work on vectors. Derivative operators take a function and return a new function. For example,

$$z = L[y] = \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y$$

- This one is linear because

$$L[cy] = cL[y]$$

$$L[y + z] = L[y] + L[z]$$

Note: y, z are functions of t and c is a constant.

Some connections to linear (matrix) algebra

- A homogeneous matrix equation has the form

$$A\overline{x} = \overline{0}$$

- A non-homogeneous matrix equation has the form

$$A\overline{x} = \overline{b}$$

- A homogeneous differential equation has the form

$$L[y] = 0$$

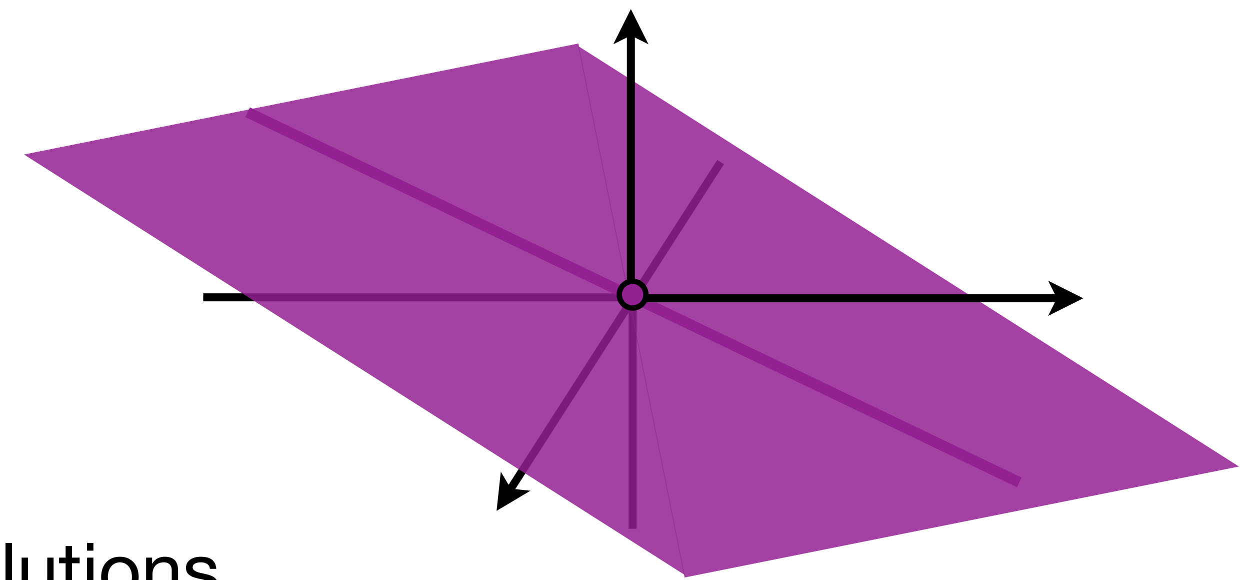
- A non-homogeneous differential equation has the form

$$L[y] = g(t)$$

Solutions to homogeneous matrix equations

- The matrix equation $A\overline{x} = \overline{0}$ could have (depending on A)

  - ☆ (A) no solutions.
  - ⇒ (B) exactly one solution.
  - ⇒ (C) a one-parameter family of solutions.
  - ⇒ (D) an n-parameter family of solutions.



Possibilities:

$$\overline{x} = \overline{0}$$

$$\overline{x} = C\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\overline{x} = C_1\begin{pmatrix}1\\-1\\1\end{pmatrix} + C_2\begin{pmatrix}0\\2\\1\end{pmatrix}$$

Choose the answer that is incorrect.

Solutions to homogeneous matrix equations

- Example 1. Solve the equation $A\overline{x} = \overline{0}$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & -2 \\ 2 & 1 & 1 \end{pmatrix}$$

Each equation describes a plane.

- Row reduction gives

$$A \sim \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{pmatrix}$$

In this case, only two of them really matter.

- so $x_1 - \frac{1}{3}x_3 = 0$ and $x_2 + \frac{5}{3}x_3 = 0$ and x_3 can be whatever (because it doesn't have a leading one).

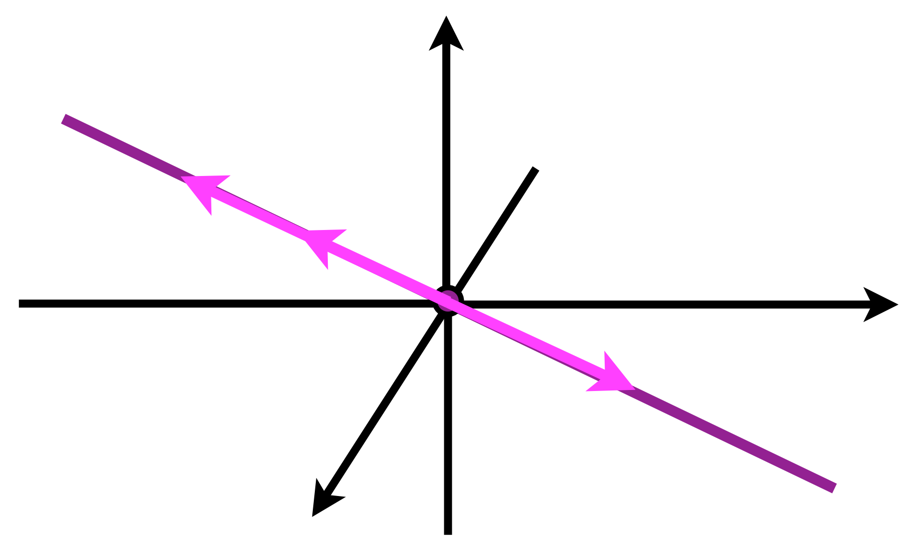
Solutions to homogeneous matrix equations

- Example 1. Solve the equation $A\overline{x} = \overline{0}$.
- so $x_1 - \frac{1}{3}x_3 = 0$ and $x_2 + \frac{5}{3}x_3 = 0$ and x_3 can be whatever.

$$x_1 = \frac{1}{3}x_3 \qquad\qquad x_1 = \frac{1}{3}C$$

$$x_2 = -\frac{5}{3}x_3 \qquad\qquad x_2 = -\frac{5}{3}C$$

$$x_3 = C$$



- Thus, the solution can be written as $\overline{x} = \cancel{\frac{C}{3}}\, C' \begin{pmatrix} \mathbf{1} \\ \mathbf{-5} \\ \mathbf{3} \end{pmatrix}$.

Solutions to homogeneous matrix equations

- Example 2. Solve the equation $A\overline{x} = \overline{0}$ where

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

- Row reduction gives

$$A \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

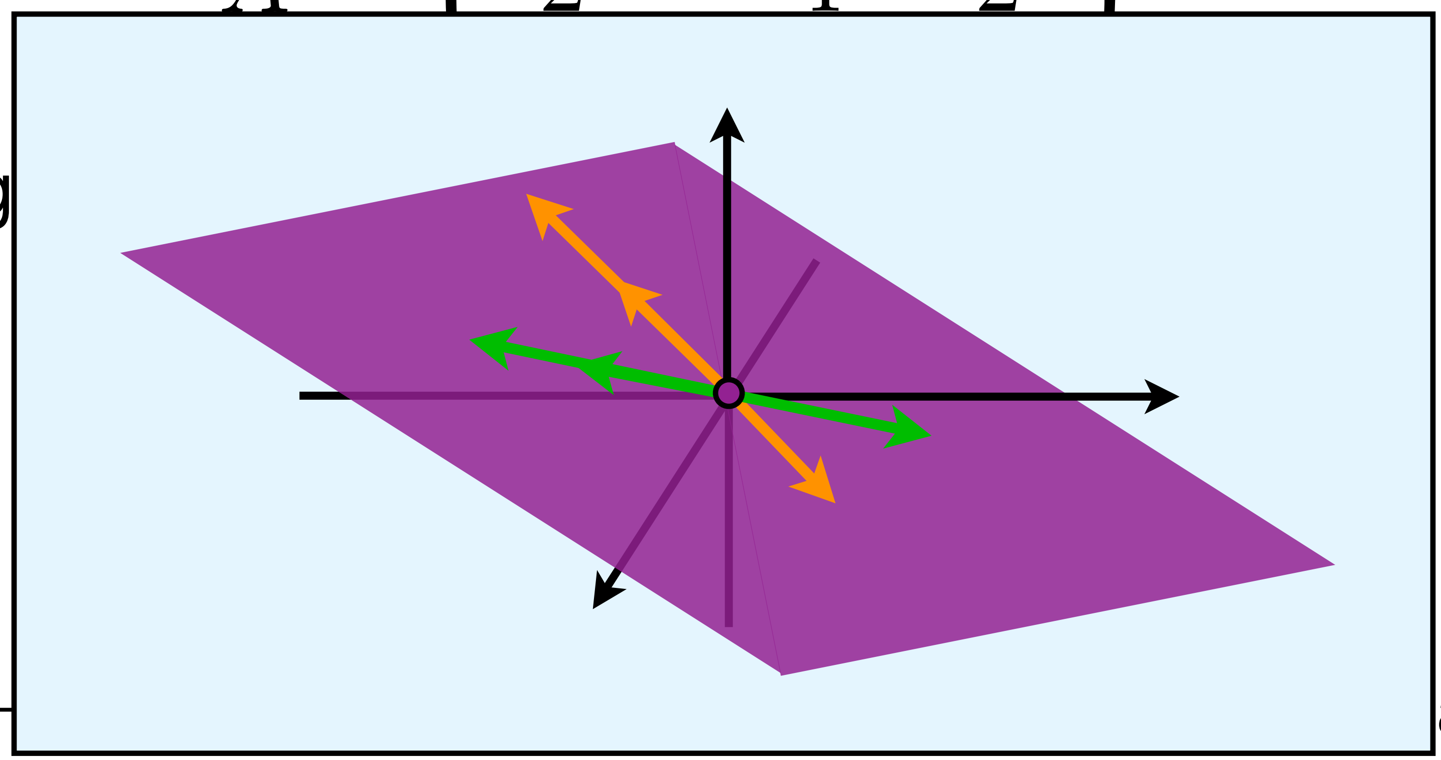
- so $x_1 - 2x_2 + x_3 = 0$ and both x_2 and x_3 can be whatever.

$$\overline{x} = C_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Solutions to homogeneous matrix equations

- Example 2. Solve the equation $A\overline{x} = \overline{0}$ where

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \end{pmatrix}$$



- Row reduction g

- so $x_1 - 2x_2 +$ atever.

$$\overline{x} = C_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Solutions to non-homogeneous matrix equations

- Example 3. Solve the equation $A\overline{x} = \overline{b}$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & -2 \\ 2 & 1 & 1 \end{pmatrix} \text{ and } \overline{b} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

- Row reduction gives

$$\left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 5/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

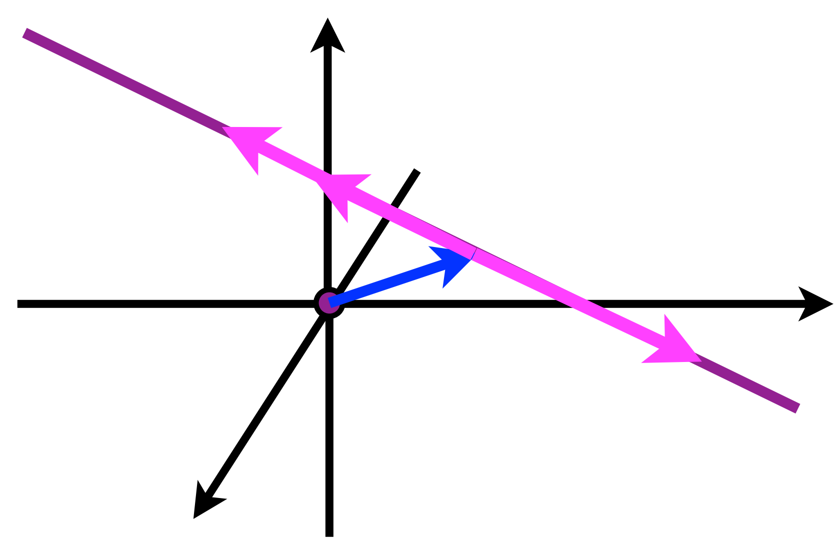
- so $x_1 - \frac{1}{3}x_3 = \frac{2}{3}$ and $x_2 + \frac{5}{3}x_3 = \frac{2}{3}$ and x_3 can be whatever.

Solutions to non-homogeneous matrix equations

- Example 3. Solve the equation $A\overline{x} = \overline{b}$.
- so $x_1 - \frac{1}{3}x_3 = \frac{2}{3}$ and $x_2 + \frac{5}{3}x_3 = \frac{2}{3}$ and x_3 can be whatever.

$$x_1 = \frac{1}{3}x_3 + \frac{2}{3} \qquad x_2 = -\frac{5}{3}x_3 + \frac{2}{3}$$

$$\overline{\boldsymbol{x}} = \frac{C'}{3}\begin{pmatrix}1\\-5\\3\end{pmatrix} + \begin{pmatrix}2/3\\2/3\\0\end{pmatrix}$$



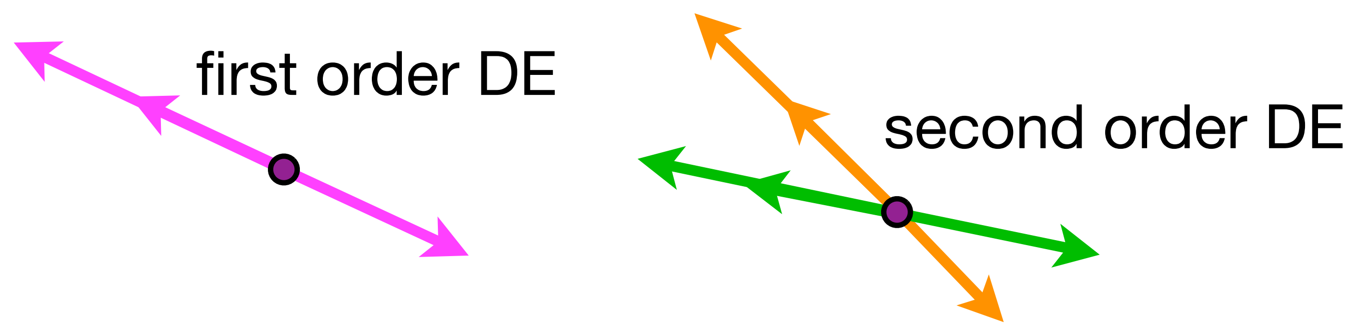
the general solution to the homogeneous problem

one particular solution to nonhomogeneous problem

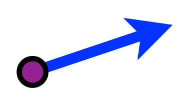
Solutions to nonhomogeneous differential equations

- To solve a nonhomogeneous differential equation:

    1. Find the general solution to the associated homogeneous problem, $y_h(t)$.

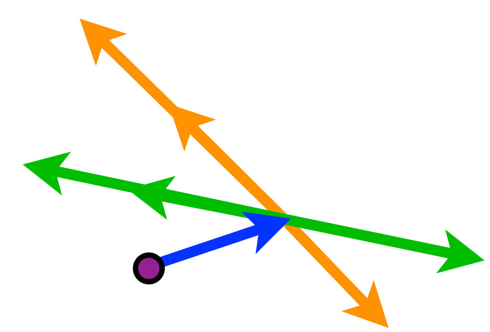


    2. Find a particular solution to the nonhomogeneous problem, $y_p(t)$.



    3. The general solution to the nonhomogeneous problem is their sum:

$$y = y_h + y_p = C_1 y_1 + C_2 y_2 + y_p$$



- For step 2, try "Method of undetermined coefficients"...

Method of undetermined coefficients (3.5)

- Example 4. Define the operator $L[y] = y'' + 2y' - 3y$. Find the general solution to $L[y] = e^{2t}$. That is, $y'' + 2y' - 3y = e^{2t}$.
  - Step 1: Solve the associated homogeneous equation
$$y'' + 2y' - 3y = 0.$$
$$y_h(t) = C_1 e^t + C_2 e^{-3t}$$
  - Step 2: What do you have to plug in to $L[\ \cdot\ ]$ to get e^{2t} out?
    - Try $y_p(t) = Ae^{2t}$.
    - $L[y_p(t)] = L[Ae^{2t}] =$
      - (A) $5e^{2t}$
      - (B) $5Ae^{2t}$ ★
      - (C) $4e^{2t}$
      - (D) $4Ae^{2t}$
    - A is an undetermined coefficient (until you determine it).

Method of undetermined coefficients (3.5)

- Example 4. Define the operator $L[y] = y'' + 2y' - 3y$. Find the general solution to $L[y] = e^{2t}$. That is, $y'' + 2y' - 3y = e^{2t}$.
  - Summarizing:
    - We know that, for any C_1 and C_2,

$$L[C_1e^t + C_2e^{-3t}] = 0$$

    - We also know that

$$L[Ae^{2t}] = 5Ae^{2t}$$

    - Finally, by linearity, we know that

$$L[C_1e^t + C_2e^{-3t} + Ae^{2t}] = 0 + 5Ae^{2t}$$

    - So what's left to do to find our general solution? Pick A =?1/5.

Method of undetermined coefficients (3.5)

- Example 5. Find the general solution to the equation $y'' - 4y = e^t$.
  - What is the solution to the associated homogeneous equation?

☆ (A) $y_h(t) = C_1 e^{2t} + C_2 e^{-2t}$

(B) $y_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$

(C) $y_h(t) = C_1 e^{2t} + C_2 t e^{2t}$

(D) $y_h(t) = C_1 \cos(2t) + C_2 \sin(2t) + e^t$

(E) Don't know.

Method of undetermined coefficients (3.5)

- Example 5. Find the general solution to the equation $y'' - 4y = e^t$.
  - What is the form of the particular solution?

    (A) $y_p(t) = Ae^{2t}$

    (B) $y_p(t) = Ae^{-2t}$

    ☆ (C) $y_p(t) = Ae^t$

    (D) $y_p(t) = Ate^t$

    (E) Don't know

Method of undetermined coefficients (3.5)

- Example 5. Find the general solution to the equation $y'' - 4y = e^t$.
  - What is the value of A that gives the particular solution (Ae^t) ?

(A) A = 1

(B) A = 3

(C) A = 1/3

☆ (D) A = -1/3

(E) Don't know.

Method of undetermined coefficients (3.5)

- Example 7. Find the general solution to the equation $y'' - 4y = e^{2t}$.
  - What is the solution to the associated homogeneous equation?

    ☆ (A) $y_h(t) = C_1 e^{2t} + C_2 e^{-2t}$

    (B) $y_h(t) = C_1 \cos(2t) + C$

    (C) $y_h(t) = C$

    $C_1 e^{2t} + C_2 e^{-2t} + e^t$

    (E) Don't know.

Same as the last example

Method of undetermined coefficients (3.5)

- Example 7. Find the general solution to the equation $y'' - 4y = e^{2t}$.
  - What is the form of the particular solution?

(A) $y_p(t) = Ae^{2t}$

(B) $y_p(t) = Ae^{-2t}$

★ (C) $y_p(t) = Ate^{2t}$

(D) $y_p(t) = Ae^{t}$

(E) $y_p(t) = Ate^{t}$

$$(Ae^{2t})'' - 4Ae^{2t} = 0 \ !$$

- Simpler example in which the RHS is a solution to the homogeneous problem.

$$y' - y = e^t$$

$$e^{-t}y' - e^{-t}y = 1$$

$$y = te^t + Ce^t$$

- General rule: when your guess at y_p makes LHS=0, try multiplying it by t.

Method of undetermined coefficients (3.5)

- Example 7. Find the general solution to the equation $y'' - 4y = e^{2t}$.
  - What is the value of A that gives the particular solution (Ate^{2t})?

(A) A = 1

(B) A = 4

(C) A = -4

☆ (D) A = 1/4

(E) A = -1/4

$$(Ate^{2t})' = Ae^{2t} + 2Ate^{2t}$$

$$(Ate^{2t})'' = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t}$$

$$= 4Ae^{2t} + 4Ate^{2t}$$

$$\left(Ate^{2t}\right)'' - 4\left(Ate^{2t}\right) = 4Ae^{2t}$$

Need: $= e^{2t}$

Method of undetermined coefficients (3.5)

- Example 8. Find the general solution to $y'' - 4y = \cos(2t)$.
  - What is the form of the particular solution?

☆ (A) $y_p(t) = A\cos(2t)$

(B) $y_p(t) = A\sin(2t)$

☆ (C) $y_p(t) = A\cos(2t) + B\sin(2t)$

(D) $y_p(t) = t(A\cos(2t) + B\sin(2t))$

(E) $y_p(t) = e^{2t}(A\cos(2t) + B\sin(2t))$

Challenge: What small change to the DE makes (D) correct?

Method of undetermined coefficients (3.5)

- Example 8. Find the general solution to $y'' + y' - 4y = \cos(2t)$.
  - What is the form of the particular solution?

(A) $y_p(t) = A\cos(2t)$

(B) $y_p(t) = A\sin(2t)$

☆ (C) $y_p(t) = A\cos(2t) + B\sin(2t)$

(D) $y_p(t) = t(A\cos(2t) + B\sin(2t))$

(E) $y_p(t) = e^{2t}(A\cos(2t) + B\sin(2t))$

Method of undetermined coefficients (3.5)

- Example 9. Find the general solution to $y'' - 4y = t^3$.
  - What is the form of the particular solution?

(A) $y_p(t) = At^3$

(B) $y_p(t) = At^3 + Bt^2 + Ct$

☆ (C) $y_p(t) = At^3 + Bt^2 + Ct + D$

(D) $y_p(t) = At^3 + Be^{2t} + Ce^{-2t}$ ← waste of time including solution to homogeneous eq.

(E) Don't know.

Method of undetermined coefficients (3.5)

- When RHS is sum of terms:

$$y'' - 4y = \cos(2t) + t^3$$

$$y_p(t) = A\cos(2t) + B\sin(2t) + Ct^3 + Dt^2 + Et + F$$

Method of undetermined coefficients (3.5)

- Example 10. Find the general solution to $y'' + 2y' = e^{2t} + t^3$.
  - What is the form of the particular solution?

(A) $y_p(t) = Ae^{2t} + Bt^3 + Ct^2 + Dt$

(B) $y_p(t) = Ae^{2t} + Bt^3 + Ct^2 + Dt + E$

☆ (C) $y_p(t) = Ae^{2t} + (Bt^4 + Ct^3 + Dt^2 + Et)$
$y_p(t) = Ae^{2t} + t(Bt^3 + Ct^2 + Dt + E)$

(D) $y_p(t) = Ae^{2t} + Be^{-2t} + Ct^3 + Dt^2 + Et + F$

(E) $y_p(t) = Ae^{2t} + Bte^{2t} + Ct^3 + Dt^2 + Et + F$

For each wrong answer, for what DE is it the correct form?

Method of undetermined coefficients (3.5)

- Example 11. Find the general solution to $y'' - 4y = t^3 e^{2t}$.
  - What is the form of the particular solution?

(A) $y_p(t) = (At^3 + Bt^2 + Ct + D)e^{2t}$

(B) $y_p(t) = (At^3 + Bt^2 + Ct)e^{2t}$

(C) $y_p(t) = (At^3 + Bt^2 + Ct)e^{2t}$
$\qquad + (Dt^3 + Et^2 + Ft)e^{-2t}$

★ (D) $y_p(t) = (At^4 + Bt^3 + Ct^2 + Dt)e^{2t}$
$y_p(t) = t(At^3 + Bt^2 + Ct + D)e^{2t}$

(E) $y_p(t) = (At^4 + Bt^3 + Ct^2 + Dt + E)e^{2t}$

Method of undetermined coefficients (3.5)

- Summary - finding a particular solution to L[y] = g(t).
  - Include all functions that are part of the g(t) family (e.g. cos **and** sin)
  - If part of the g(t) family is a solution to the homogeneous (h-)problem, use t x (g(t) family).
  - If t x (part of the g(t) family), is a solution to the h-problem, use t^2 x (g(t) family).
  - For sums, group terms into families and include a term for each.
  - For products of families, use the above rules and multiply them.
  - If your guess includes a solution to the h-problem, you may as well remove it as it won't survive L[ ] so you won't be able to determine its undetermined coefficient.