Today

- Midterm 2 comments
- Pre-lecture week 12 comments
- Post-lecture week 11 comments
- Using FS to solve the Diffusion equation.

Solving the Diffusion equation using FS - Preview

- The Diffusion equation is solved by functions of the form

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

$$c(x,t) = be^{-w^2Dt}\sin(wx)$$
$$d(x,t) = ae^{-w^2Dt}\cos(wx)$$
$$g(x,t) = \text{constant}$$

- Boundary conditions determine which of these to use.
  - For Dirichlet BCs, use c(x,t) with w = n π x / L.

$$c(0,t) = 0,\ \ c(L,t) = 0 \ \Rightarrow c_n(x,t) = b_n e^{-\frac{n^2\pi^2}{L^2}Dt}\sin\left(\frac{n\pi x}{L}\right)$$

  - For Neumann BCs, use d(x,t) with w = n π x / L and g(x,t).

$$\frac{\partial c}{\partial x}(0,t) = 0,\ \ \frac{\partial c}{\partial x}(L,t) = 0 \ \Rightarrow d_n(x,t) = a_n e^{-\frac{n^2\pi^2}{L^2}Dt}\cos\left(\frac{n\pi x}{L}\right)$$

- The initial condition determines the a_n or b_n values via Fourier series.

The Diffusion Equation

- What does a steady state of the Diffusion equation look like?

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

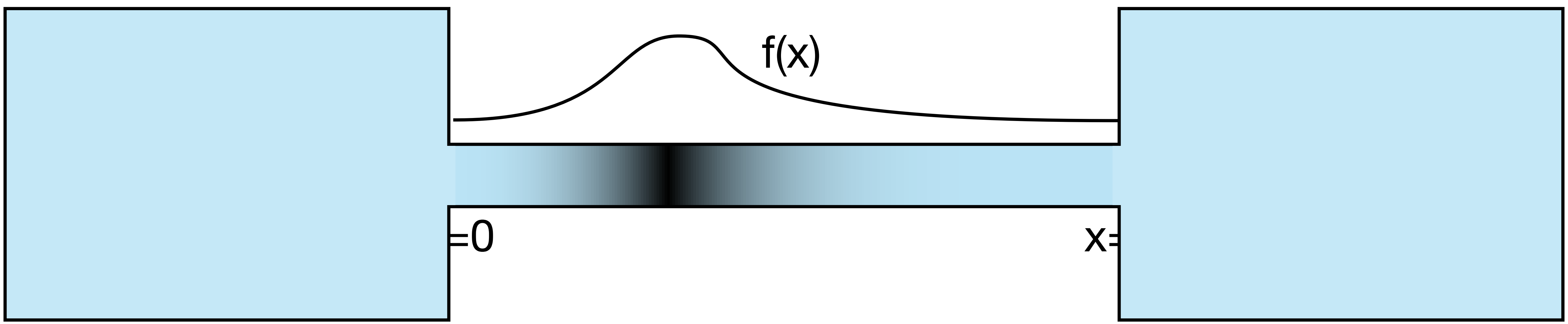
$$D\frac{d^2c}{dx^2} = 0$$

$$\frac{dc}{dx} = A$$

$$c_{ss}(x) = Ax + B$$

The Diffusion Equation

An initial condition specifies where all the mass is initially: $c(x,0) = f(x)$.



A common boundary condition (Dirichlet) states that the concentration is forced to be zero at the end point(s) (infinite reservoir):

$$c(0,t) = 0,\ \ c(L,t) = 0$$

What is the steady state in this case? $c_{ss}(x)=Ax+B$

$c_{ss}(0)=B=0$, $c_{ss}(L)=AL=0$, $A=0$, $B=0$ so $c_{ss}(x)=0$!

The Diffusion Equation

Solve $$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

subject to $c(0,t) = 0,\ c(L,t) = 0$ (Dirichlet boundary conditions)

and $c(x,0) = f(x)$ (initial condition)

For any a and any w, the following are both solutions to the PDE:

$$c(x,t) = be^{-w^2Dt}\sin(wx)$$

~~$d(x,t) = ae^{-w^2Dt}\cos(wx)$~~

$$c(0,t) = b\sin(0) = 0$$ Yup!

$$d(0,t) = a\cos(0) = a$$

$$c(L,t) = b\sin(wL) = 0$$

$$wL = n\pi$$

$$w = \frac{n\pi}{L}$$

For any n and any b_n,

$$c_n(x,t) = b_n e^{-\frac{n^2\pi^2}{L^2}Dt}\sin\left(\frac{n\pi}{L}x\right)$$

solves the PDE and BCs. What about IC?

The Diffusion Equation

So far, we can add these up with any choice of b_n (provided the series converges) to get a solution.

$$c(x,t) = \sum_{n=1}^{\infty} c_n(x,t) \; = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{L^2}Dt} \sin\left(\frac{n\pi}{L}x\right)$$

Now, choose b_n so that c(x,t) satisfies the IC. That is, c(x,0)=f(x):

$$c(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

That means we choose b_n to be the coefficients of a Fourier series for f(x) consisting entirely of sin terms.

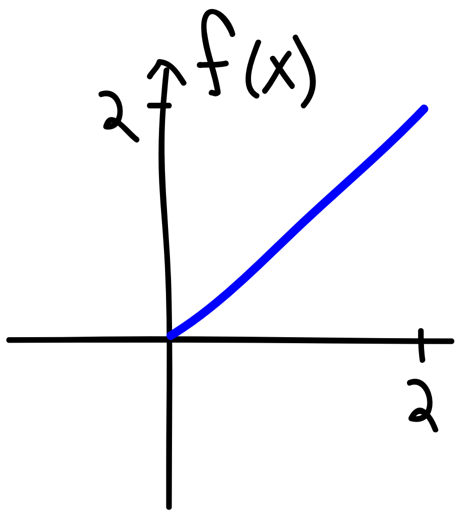
How do we get a Fourier sine series for f(x) defined on [0,L]?

The Diffusion Equation

Solve the equation $\dfrac{dc}{dt} = D\dfrac{d^2c}{dx^2}$

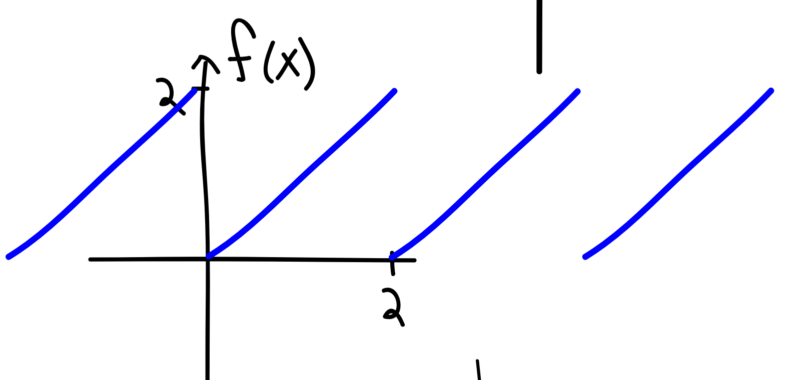
subject to boundary conditions $c(0,t) = 0, \ c(2,t) = 0$

and initial condition $c(x,0) = x$ defined on [0,2].

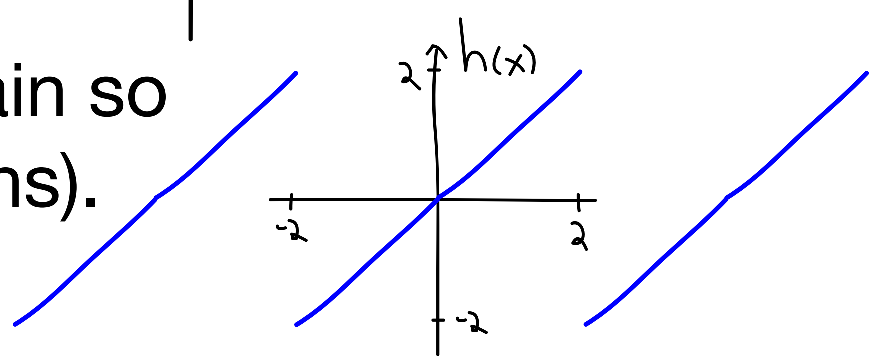


How do we solve this?

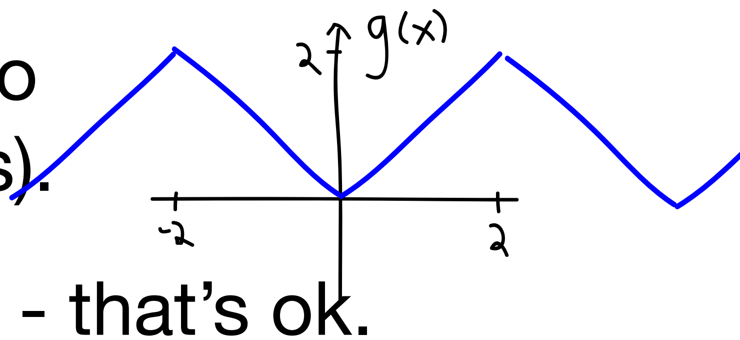
(A) Extend IC so it's periodic (P=2), then find FS.



(B) Extend IC so it's odd on [-2,2], then extend again so it's periodic (P=4), finally find FS (all sin functions).



(C) Extend IC so it's even on [-2,2], then extend again so it's periodic (P=4), finally find FS (const. and cosines).



Note: the IC does not satisfy the BC at x=L in this case - that's ok.

The Diffusion equation - BC terminology

$$c(0,t) = 0, \;\; c(L,t) = 0$$

Dirichlet BCs, huge empty chambers at both ends of the pipe.

- use sin functions for Fourier series (Fourier Sine series)
- extend f(x) as an odd function on [-L,L] and then extend as periodic (2L).

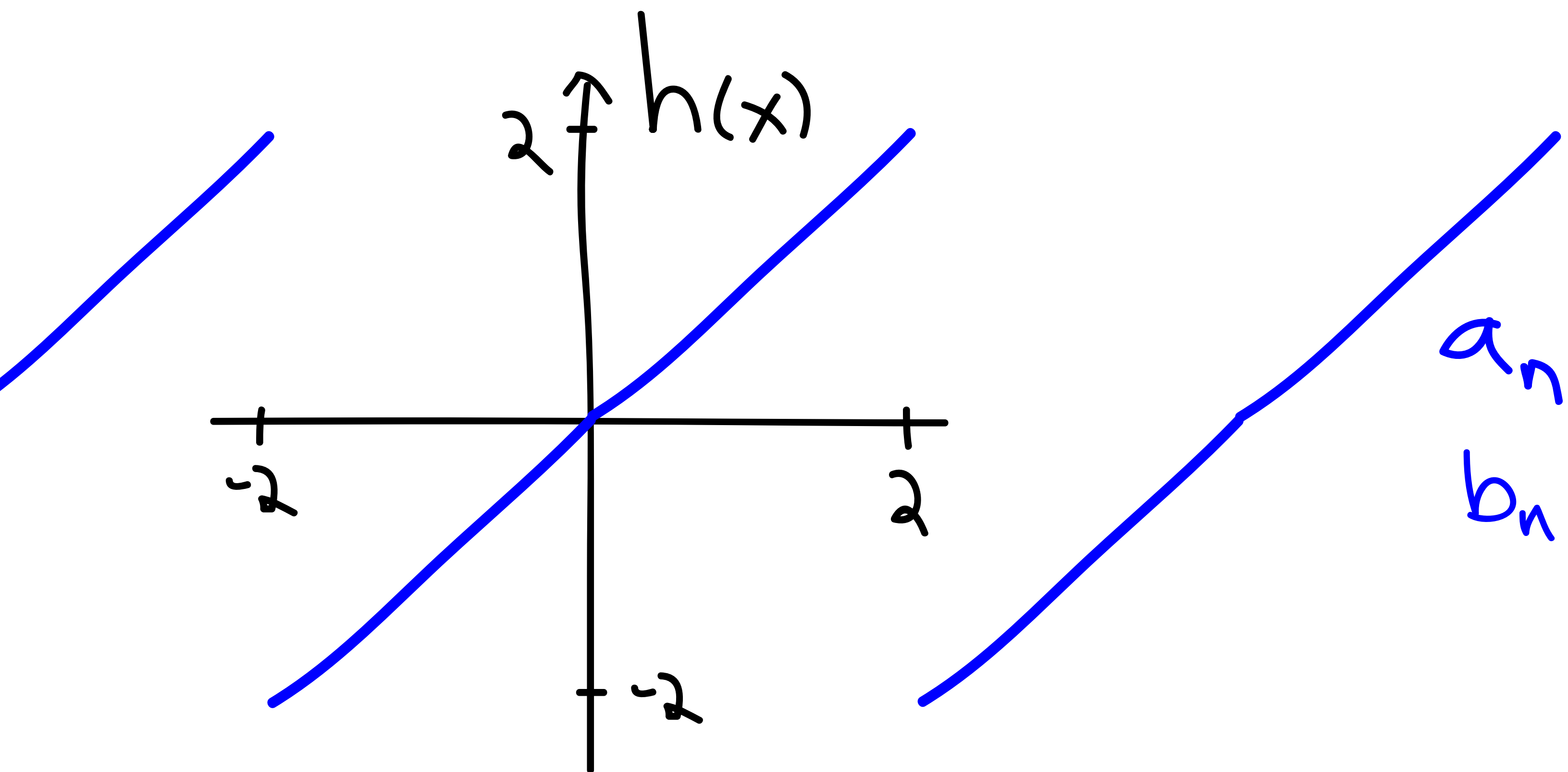
$$\frac{\partial c}{\partial x}(0,t) = 0, \;\; \frac{\partial c}{\partial x}(L,t) = 0$$

Neumann BCs, both ends of the pipe are sealed so nothing escapes.

- use constant + cosine functions for Fourier series (Fourier Cosine series)
- extend f(x) as an even function on [-L,L] and then extend as periodic (2L).
- often called no flux BCs, because

$$J_0 = -D\frac{\partial c}{\partial x}(0,t) = 0 \text{ and } \;\; J_L = -D\frac{\partial c}{\partial x}(L,t) = 0$$

Examples - odd periodic extension



$a_n = 0$

$b_n = \frac{(-1)^{n+1} 4}{n\pi}$

$h(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2}$

for $x \neq -2, 2$.

What is L? L=2

$$c(x,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{n^2 \pi^2 D t}{4}} \sin\left(\frac{n\pi x}{2}\right)$$

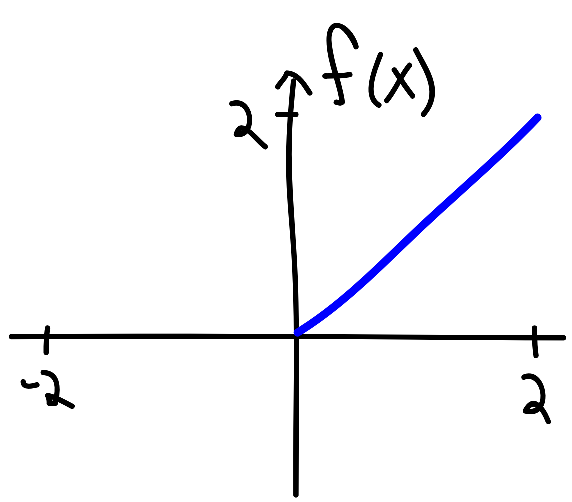
Even and odd extensions

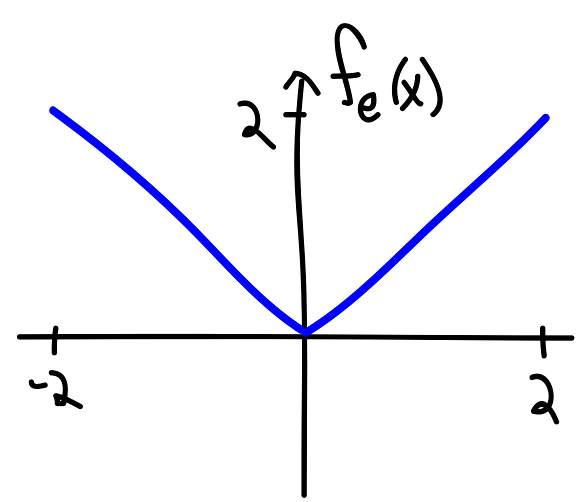
- For a function *f(x)* defined on [0,L], the even extension of *f(x)* is the function

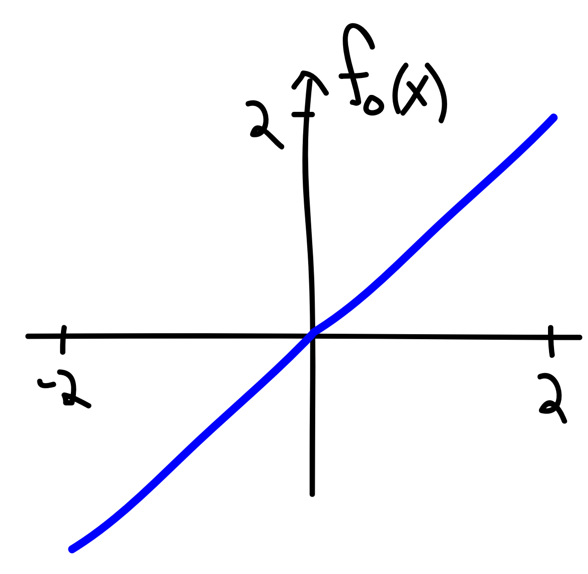
$$f_e(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq L, \\ f(-x) & \text{for } -L \leq x < 0. \end{cases}$$

- For a function *f(x)* defined on [0,L], the odd extension of *f(x)* is the function

$$f_o(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq L, \\ -f(-x) & \text{for } -L \leq x < 0. \end{cases}$$







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- Because these functions are even/odd, their Fourier Series have a couple simplifying features:

$$f_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

The Diffusion equation

Solve the equation $\dfrac{dc}{dt} = D\dfrac{d^2c}{dx^2}$

subject to boundary conditions $\dfrac{\partial c}{\partial x}(0,t) = 0,\ \ \dfrac{\partial c}{\partial x}(2,t) = 0$

and initial condition $c(x,0) = x$ defined on [0,2].

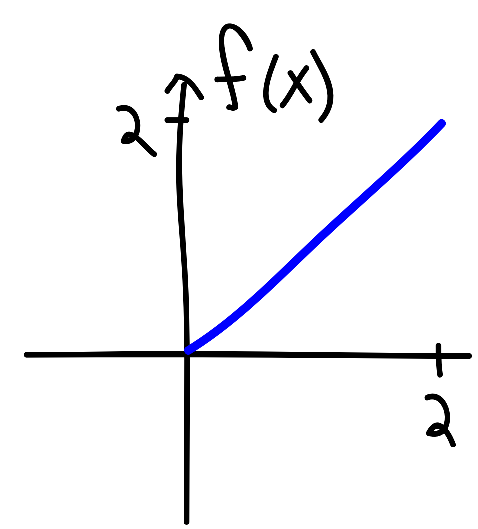
What is the steady state in this case? $c_{ss}(x) = Ax+B$

BC says A=0. B=?

$$\text{Total initial mass} = \int_0^L c(x,0)\ dx$$

$$\text{Total ``final'' mass} = \int_0^L c_{ss}(x)\ dx$$

} No flux BCs so these must be equal.



In this case, the Fourier series tells us the answer:

$$c(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{4}Dt}\cos\left(\frac{n\pi x}{2}\right) \longrightarrow c_{ss}(x)=a_0/2$$

The Diffusion equation

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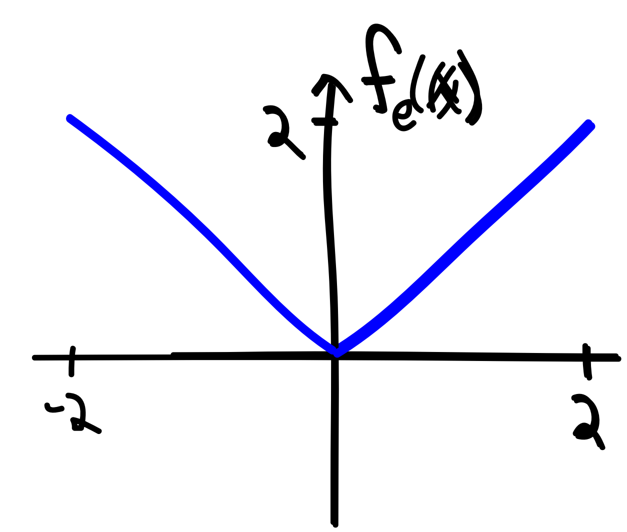
and initial condition $c(x,0) = x$ defined on [0,2].

$$c(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{4}Dt} \cos\left(\frac{n\pi x}{2}\right)$$

$$c(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) = x$$



The Diffusion equation

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and initial condition $c(x,0) = x$ defined on [0,2].

$$a_0 = 2$$

$$a_n = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{((-1)^n-1)}{n^2} \cos\frac{n\pi x}{2}$$

$$c(x,t) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{((-1)^n-1)}{n^2} e^{-\frac{n^2\pi^2}{4}Dt} \cos\frac{n\pi x}{2}$$

Solving the Diffusion equation using FS - Preview

- The Diffusion equation ties the exponent to the frequency:

$$\frac{dc}{dt} = D\frac{d^2c}{dx^2}$$

$$c(x,t) = be^{-w^2Dt}\sin(wx)$$
$$d(x,t) = ae^{-w^2Dt}\cos(wx)$$
$$g(x,t) = \text{constant}$$

- Boundary conditions whether you need a Fourier sine or cosine series and determines the frequency ω.

$$c(0,t) = 0,\ \ c(L,t) = 0\ \Rightarrow c_n(x,t) = b_n e^{-\frac{n^2\pi^2}{L^2}Dt}\sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial c}{\partial x}(0,t) = 0,\ \ \frac{\partial c}{\partial x}(L,t) = 0\ \Rightarrow d_n(x,t) = a_n e^{-\frac{n^2\pi^2}{L^2}Dt}\cos\left(\frac{n\pi x}{L}\right)$$

- The initial condition determines the a_n values via Fourier series.

$$c(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \quad \text{or} \quad c(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) = f(x)$$