Today

- Summary of 2x2 systems with constant coefficients.
- Nonhomogeneous example

Summary - homogeneous 2x2 systems

- For the system of equations $\mathbf{x}' = A\mathbf{x}$, we always have $\mathbf{x}(t) = \mathbf{0}$ as a steady state solution.
- If *A* is singular matrix with $A\mathbf{v} = \mathbf{0}$ then $\mathbf{x}(t) = \mathbf{v}$ is also a steady state solution. In fact, $\mathbf{x}(t) = c\mathbf{v}$ is a steady state for all *c*. It is also an eigenvector associated with eigenvalue $\lambda = 0$.
- If *A* is nonsingular then $\mathbf{x}(t) = \mathbf{0}$ is the only steady state.
- Quick way to determine how all other solutions behave:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

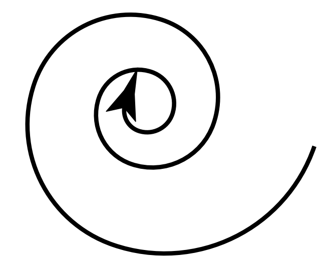
$$\det(A - \lambda I) = \det\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc$$

$$= \lambda^2 - (a + d)\lambda + ad - bc = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

Summary - homogeneous 2x2 systems

- When do the solutions spiral in to the origin?

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$



(A) $\begin{cases} \mathrm{tr}A < 0 \\ (\mathrm{tr}A)^2 < 4\det A \end{cases}$

(B) $\begin{cases} \mathrm{tr}A > 0 \\ (\mathrm{tr}A)^2 < 4\det A \end{cases}$

(C) $\begin{cases} \mathrm{tr}A < 0 \\ (\mathrm{tr}A)^2 > 4\det A \end{cases}$

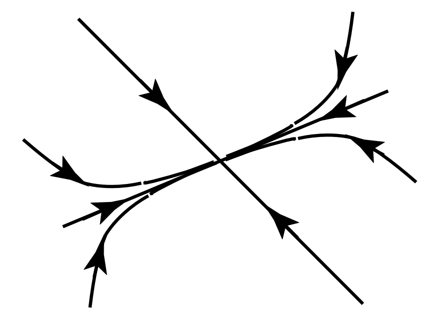
(D) $\begin{cases} \mathrm{tr}A > 0 \\ (\mathrm{tr}A)^2 > 4\det A \end{cases}$

(E) Explain, please.

Summary - homogeneous 2x2 systems

- When is the origin a stable node?

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$



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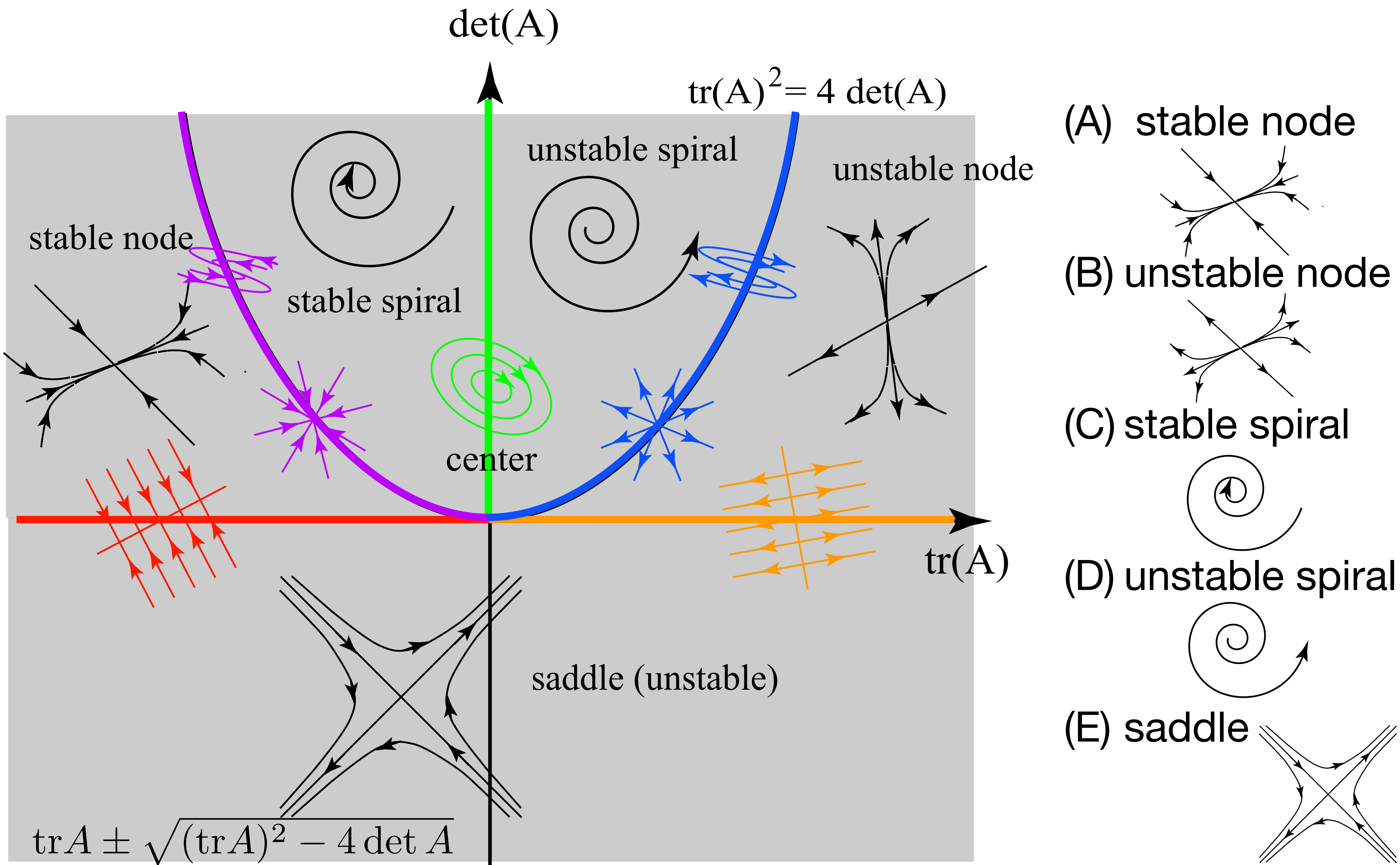
Summary - homogeneous 2x2 systems

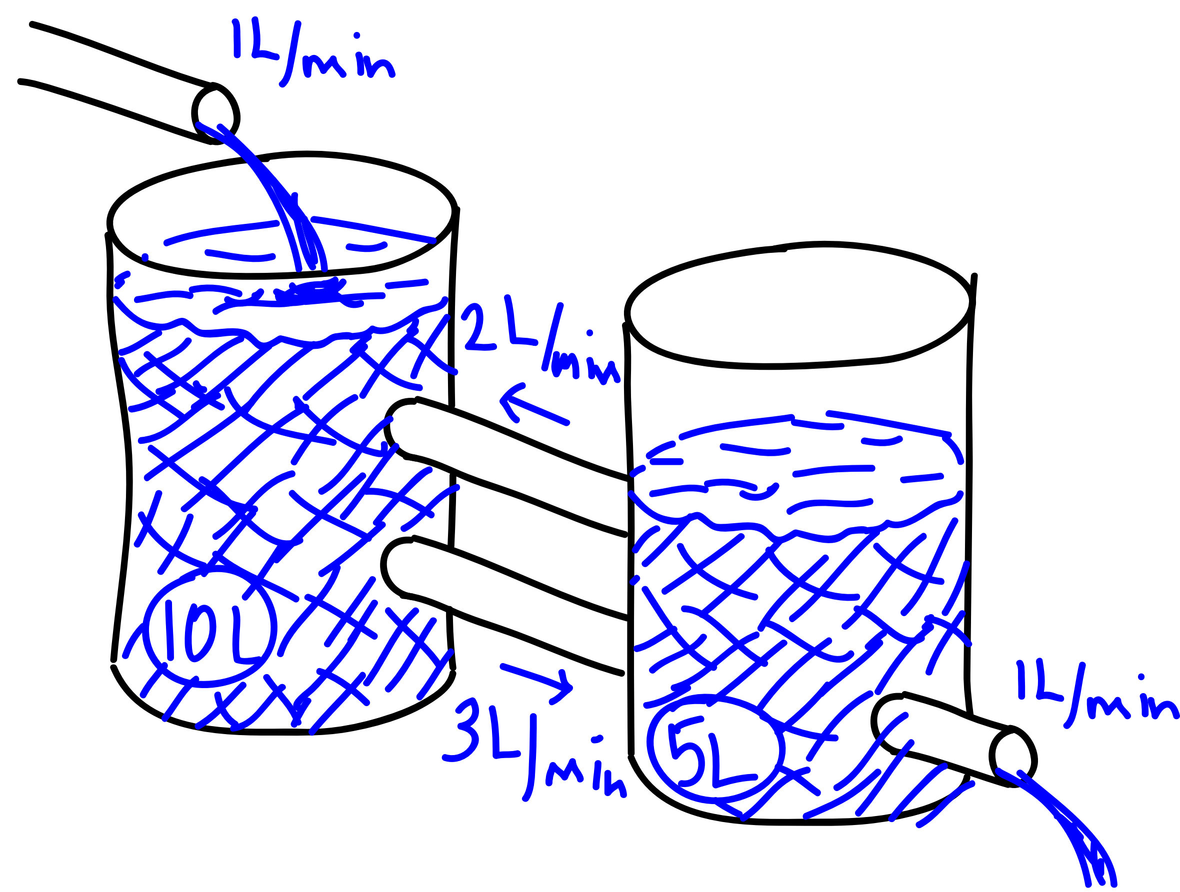


$$\lambda = \frac{\mathrm{tr} A \pm \sqrt{(\mathrm{tr} A)^2 - 4 \det A}}{2}$$

Nonhomogeneous case - example

- Salt water flows into a tank holding 10 L of water at a rate of 1 L/min with a concentration of 200 g/L. The well-mixed solution flows from that tank into a tank holding 5 L through a pipe at 3 L/min. Another pipe takes the solution in the second tank back into the first at a rate of 2 L/min. Finally, solution drains out of the second tank at a rate of 1 L/min.
- Write down a system of equations in matrix form for the mass of salt in each tank.

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}' = \begin{pmatrix} -\frac{3}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} 200 \\ 0 \end{pmatrix}$$



Nonhomogeneous case - example

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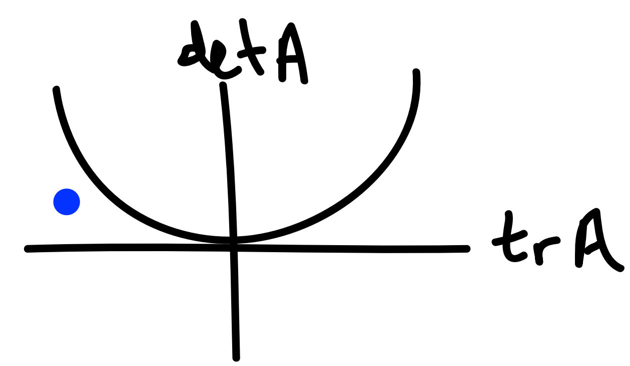
- Solve the system of equations.

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}' = \begin{pmatrix} -\frac{3}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} 200 \\ 0 \end{pmatrix}$$

$$\mathrm{tr}A = -\frac{9}{10} \qquad (\mathrm{tr}A)^2 = \frac{81}{100}$$

$$\det A = \frac{9}{50} - \frac{6}{50} = \frac{3}{50} \qquad 4\det A = \frac{12}{50}$$



Both evalues negative!

Nonhomogeneous case - example

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$$\mathrm{tr} A = -\frac{9}{10} \qquad (\mathrm{tr} A)^2 = \frac{81}{100}$$

$$\det A = \frac{9}{50} - \frac{6}{50} = \frac{3}{50} \qquad 4 \det A = \frac{12}{50} = \frac{24}{100}$$

$$\mathbf{m_h}(t) = C_1 e^{\lambda_1 t}\mathbf{v_1} + C_2 e^{\lambda_2 t}\mathbf{v_2} \qquad \left(\lambda_{1,2} = -\frac{9}{20} \pm \frac{\sqrt{57}}{20} \right)$$

$$\mathbf{m_p}(t) = \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\mathbf{0} = A\mathbf{w} + \begin{pmatrix} 200 \\ 0 \end{pmatrix} \rightarrow A\mathbf{w} = -\begin{pmatrix} 200 \\ 0 \end{pmatrix} \rightarrow \mathbf{w} = \begin{pmatrix} 2000 \\ 1000 \end{pmatrix}$$

Nonhomogeneous case - example

- A “Method of undetermined coefficients” similar to what we saw for second order equations can be used for systems.
- For this course, I’ll only test you on constant nonhomogeneous terms (like the previous example) in the context of mixing problems.

Laplace transforms

- Motivation for Laplace transforms:
  - We know how to solve $ay'' + by' + cy = g(t)$ when $g(t)$ is polynomial, exponential, trig.
  - In applications, $g(t)$ is often "piece-wise continuous" meaning that it consists of a finite number of pieces with jump discontinuities in between. For example,

$$g(t) = \begin{cases} \sin(\omega t) & 0 < t < 10, \\ 0 & t \geq 10. \end{cases}$$

  - These can be handled by previous techniques (modified) but it isn't pretty.