Today

- Introduction to systems of equations
- Direction fields
- Eigenvalues and eigenvectors
- Finding the general solution (distinct e-value case)

Introduction to systems of equations

- So far, we've only dealt with equations with one unknown function. Sometimes, we'll be interested in more than one unknown function.
- Examples:
  - position of object in one dimensional space in terms of x, v:

$$mx'' + \gamma x' + kx = 0 \;\rightarrow\; mv' + \gamma v + kx = 0$$

$$x' = v \qquad\qquad v' = -\frac{\gamma}{m}v - \frac{k}{m}x$$

$$x'' = v'$$

$$\begin{aligned} x' &= v \\ v' &= -\frac{k}{m}x - \frac{\gamma}{m}v \end{aligned}$$

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

Introduction to systems of equations

- So far, we've only dealt with equations with one unknown function. Sometimes, we'll be interested in more than one unknown function.
- Examples:
    - position of object in one dimensional space in terms of x, v.
    - position of an object in a plane (x, y coordinates) or three dimensional space (x, y, z coordinates).
    - positions of multiple objects (two or more masses linked by springs).
    - concentration in connected chambers (saltwater in multiple tanks, intracellular and extracellular, blood stream and organs).
    - populations of two species (e.g. predator and prey).

Introduction to systems of equations

- As with single equations, we have linear and nonlinear systems:

$$\frac{dx}{dt} = t^2 x - y + \cos(2t)$$
$$\frac{dy}{dt} = x + 4\sin(t)y + t^3$$

$$\frac{dx}{dt} = t^2 x - y^2$$
$$\frac{dy}{dt} = \sqrt{x} - y$$

- And we also have nonhomogeneous and homogeneous systems.

$$\frac{dx}{dt} = t^2 x - y + \cos(2t)$$
$$\frac{dy}{dt} = x + 4\sin(t)y + t^3$$

$$\frac{dx}{dt} = t^2 x - y$$
$$\frac{dy}{dt} = x + 4\sin(t)y$$

Introduction to systems of equations

- Any linear system can be written in matrix form:

$$\frac{dx}{dt} = t^2 x - y + \cos(2t)$$
$$\frac{dy}{dt} = x + 4\sin(t) y + t^3$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t^2 & -1 \\ 1 & 4\sin(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos(2t) \\ t^3 \end{pmatrix}$$

- We'll focus on the case in which the matrix has constant entries. And homogeneous. For example,

$$L\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} t^2 & -1 \\ 1 & 4\sin(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(2t) \\ t^3 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

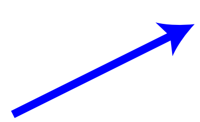
Introduction to systems of equations

- Geometric interpretation - direction fields.

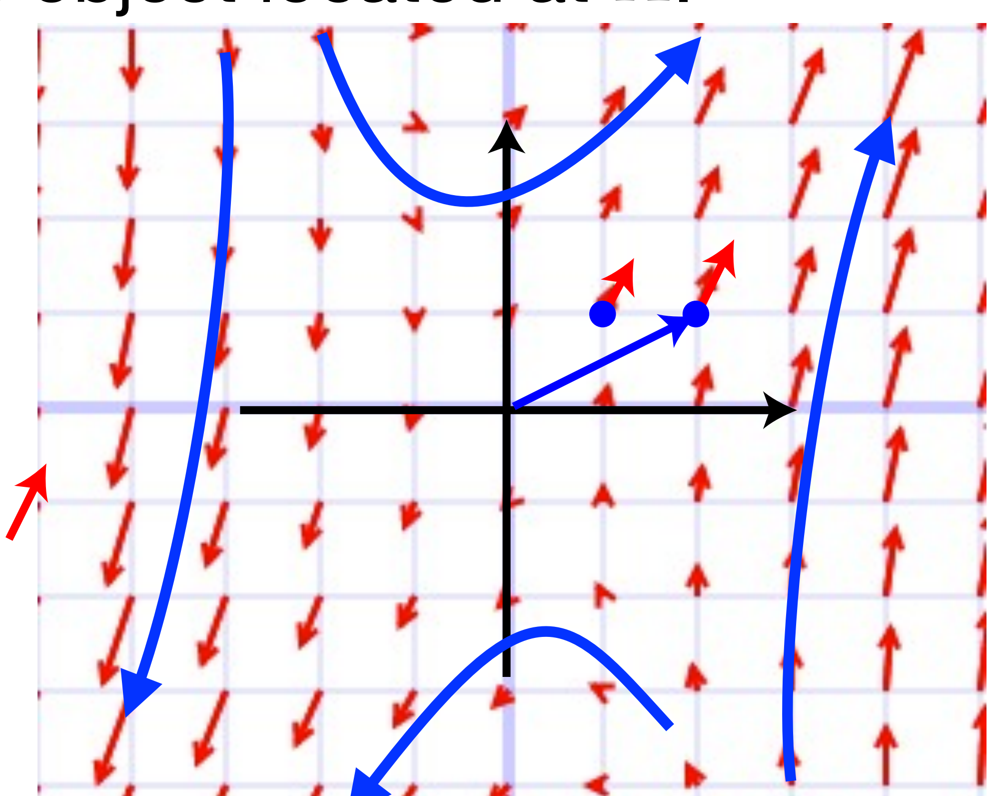
$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{or} \qquad \mathbf{x}' = A\mathbf{x}$$

- Think of the unknown functions as coordinates $(x(t), y(t))$ of an object in the plane.
- $A\mathbf{x}$ gives the velocity vector of the object located at $\mathbf{x}$.

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



$$A\mathbf{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$



- Solutions must follow the arrows.

Introduction to systems of equations

- Which of the following equations matches the given direction field?

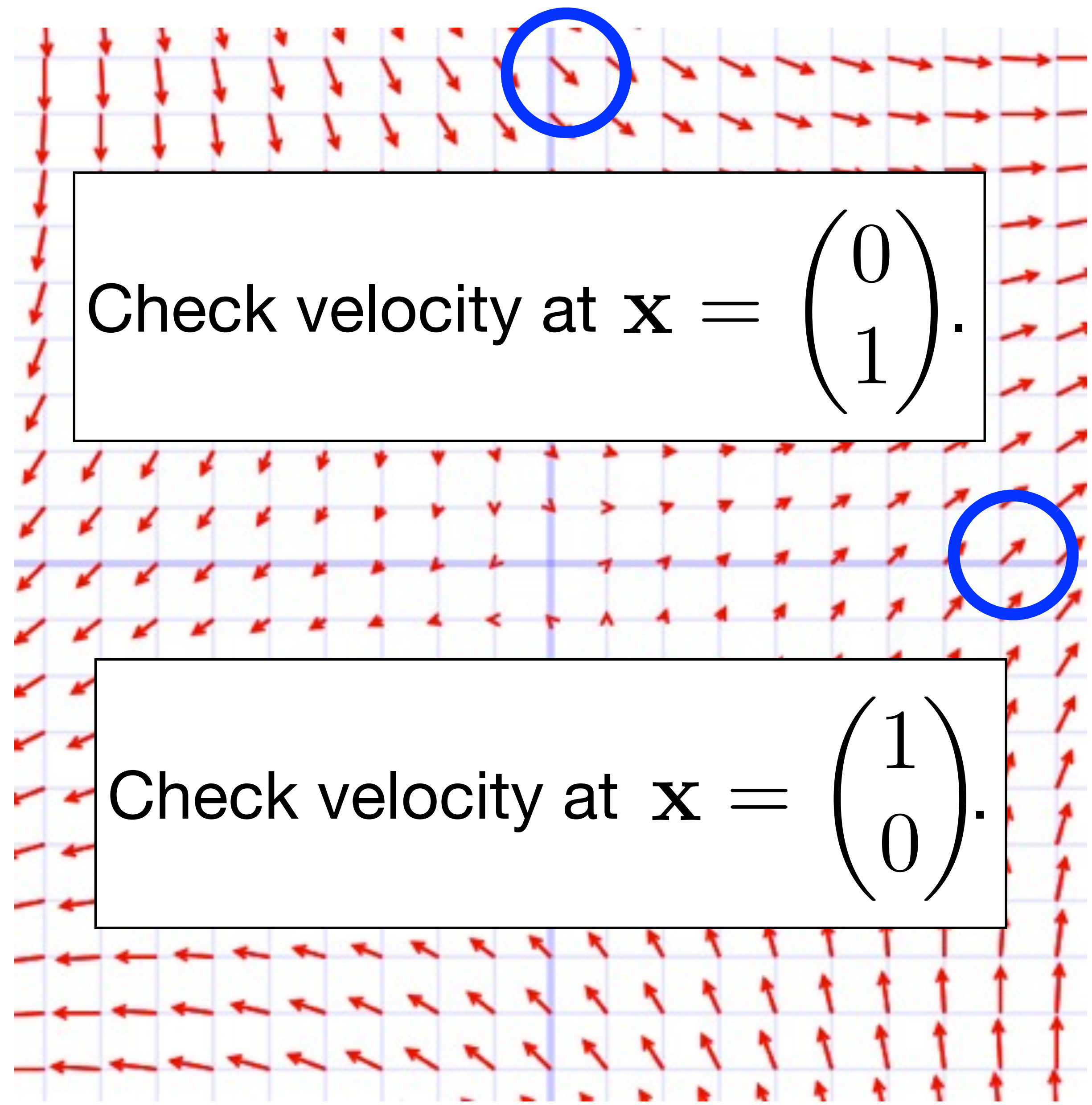
(A) $\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(B) $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(C) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

★ (D) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(E) Explain, please.



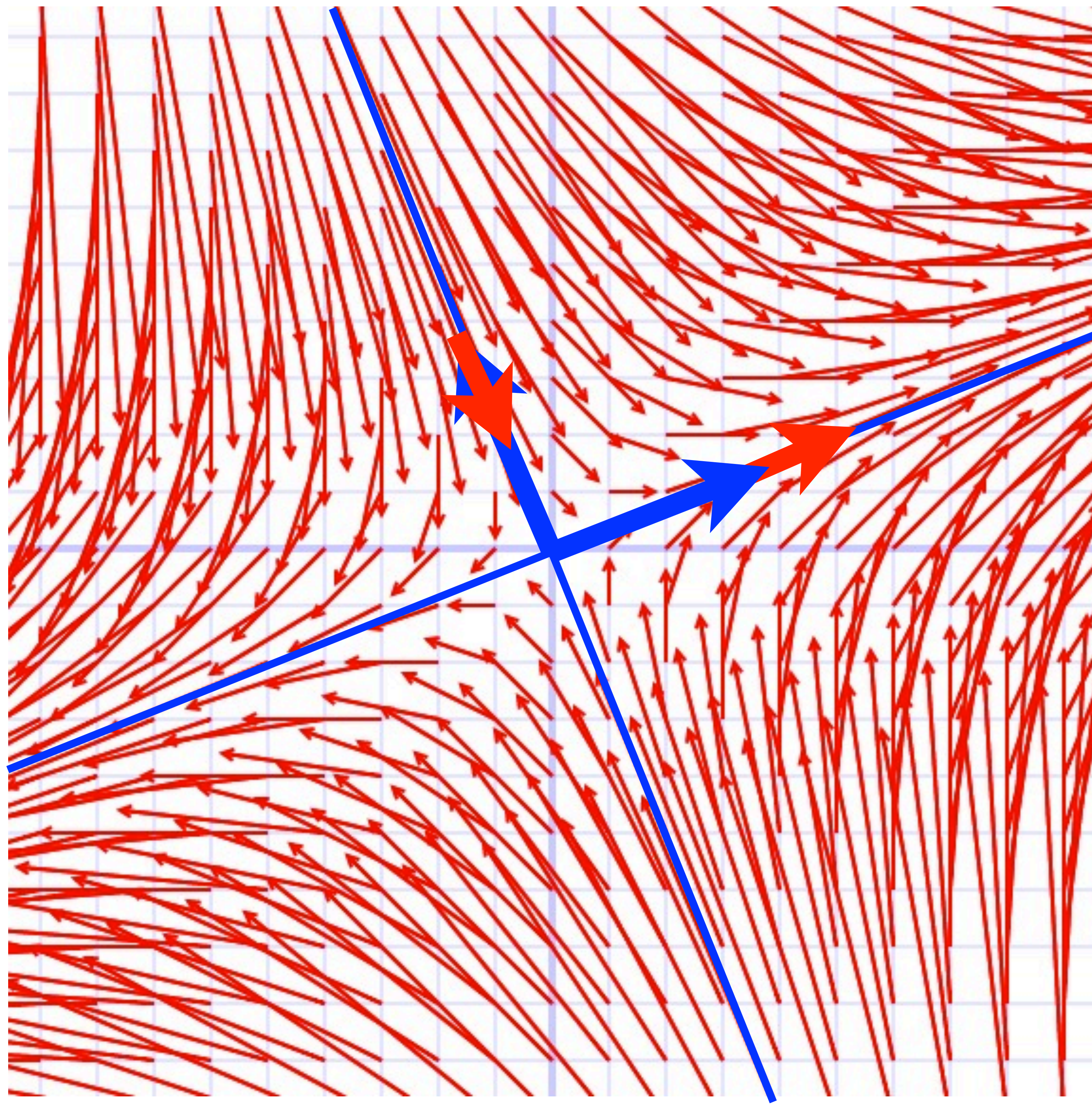
http://kevinmehall.net/p/equationexplorer/vectorfield.html#(x+y)i+(x-y)j%7C%5B-10,10,-10,10%5D

Introduction to systems of equations

- You should see two "special" directions.
- What are they?
- Directions along which the velocity vector is parallel to the position vector.
- That is, $A\mathbf{v} = \lambda\mathbf{v}$.

$$\lambda_1 = \sqrt{2}, \quad \lambda_2 = -\sqrt{2}$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1-\sqrt{2} \\ 1 \end{pmatrix}$$



Matrix review (eigen-calculations)

- Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.
- Looking for values λ and vectors $\mathbf{v}$ for which $A\mathbf{v} = \lambda\mathbf{v}$.
- What are the eigenvalues of A?

(A) 1 and -3

(B) -1 and 3

(C) 1 and 3

(D) -1 and -3

(E) Explain, please.

Matrix review (eigen-calculations)

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- Looking for values λ and vectors $\mathbf{v}$ for which $A\mathbf{v} = \lambda\mathbf{v}$.

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)^2 - 4 = 0$$

$$(\lambda^2 - 2\lambda - 3 = 0)$$

$$\lambda = 1 \pm 2 = -1, 3$$

- What are the eigenvectors associated with λ_1=-1?

(A) $\mathbf{v_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

☆(B) $\mathbf{v_1} = c\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(C) $\mathbf{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(D) $\mathbf{v_1} = c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(E) Explain, please.

Matrix review (eigen-calculations)

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$$\lambda = 1 \pm 2 = -1, 3$$

$$\lambda_1 = -1$$

$$(A + I)\mathbf{v_1} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \mathbf{v_1} = 0$$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$2v_1 + v_2 = 0$$

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

(and any scalar multiple of it)

Matrix review (eigen-calculations)

- Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.
- Looking for values λ and vectors $\mathbf{v}$ for which $A\mathbf{v} = \lambda\mathbf{v}$.

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$$\lambda = 1 \pm 2 = -1, 3$$

$$\lambda_1 = -1$$

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = 3$$

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- How do we use eigenvalues and eigenvectors to construct a general solution?

Solving a system of ODEs

- The following is a shortcut approach for 2x2 systems, mostly for insight.
- Find the general solution to the system of equations

$$x_1' = x_1 + x_2$$
$$x_2' = 4x_1 + x_2$$

or equivalently

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

- Convert this into a second order equation in only one unknown (x_1):

$$x_1'' = x_1' + x_2' = x_1' + 4x_1 + x_2$$

$$x_2 = x_1' - x_1$$

$$x_1'' = x_1' + 4x_1 + x_1' - x_1$$

$$x_1'' - 2x_1' - 3x_1 = 0$$

Solving a system of ODEs

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- Find the general solution to the system of equations

$$x_1' = x_1 + x_2 \qquad \text{or equivalently} \qquad \mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$
$$x_2' = 4x_1 + x_2$$

- Convert this into a second order equation in only one unknown (x_1):

$$x_1'' - 2x_1' - 3x_1 = 0 \quad \rightarrow r^2 - 2r - 3 = 0$$

$$r = -1, 3$$

$$x_1 = C_1 e^{-t} + C_2 e^{3t}$$

$$x_2 = x_1' - x_1 = -C_1 e^{-t} + 3C_2 e^{3t} - C_1 e^{-t} - C_2 e^{3t}$$

$$= -2C_1 e^{-t} + 2C_2 e^{3t}$$

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$$x_1 = C_1 e^{-t} + C_2 e^{3t}$$

$$x_2 = -2C_1 e^{-t} + 2C_2 e^{3t}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- Recall:

$$\lambda_1 = -1 \qquad \mathbf{v_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = 3 \qquad \mathbf{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solving a system of ODEs

- You can use the second order trick for 2x2 but in general,
  - Find eigenvalues and eigenvectors of A,
  - Assemble general solution by summing up terms of the form

$$C_n e^{\lambda_n t} \mathbf{v_n}$$

  - This works when eigenvalues are distinct or, if there are repeated eigenvalues still giving N independent eigenvectors.
  - Other cases (not enough e-vectors or complex e-values) next class.