Today

- Modeling with delta-function forcing (tanks, springs)
- Convolution
- Transfer functions

Delta-function forcing (6.5)

- Water with c_{in} = 2 g/L of sugar enters a tank at a rate of r = 1 L/min. The initially sugar-free tank holds V = 5 L and the contents are well-mixed. Water drains from the tank at a rate r. At t_{cube} = 3 min, a sugar cube of mass m_{cube} = 3 g is dropped into the tank.
    - Sketch the mass of salt in the tank as a function of time (from intuition).
    - Write down an ODE for the mass of sugar in the tank as a function of time.

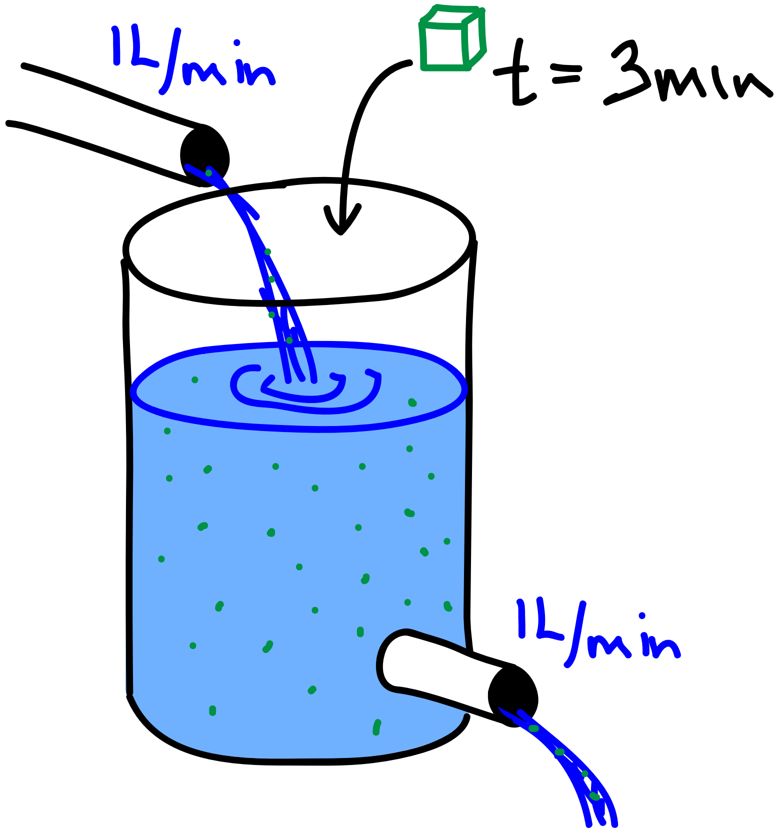
$$m' = rc_{in} - \frac{r}{V}m + m_{cube}\delta(t - t_{cube})$$

$$m' = 2 - \frac{1}{5}m + 3\ \delta(t-3)$$

    - Solve the ODE.

$$m(t) = 10(1 - e^{-t/5}) + 3u_3(t)e^{-(t-3)/5}$$

    - Sketch the solution to the ODE. How would it differ if t_{cube}=10 min?



- Note: δ(t) has units of 1/time.

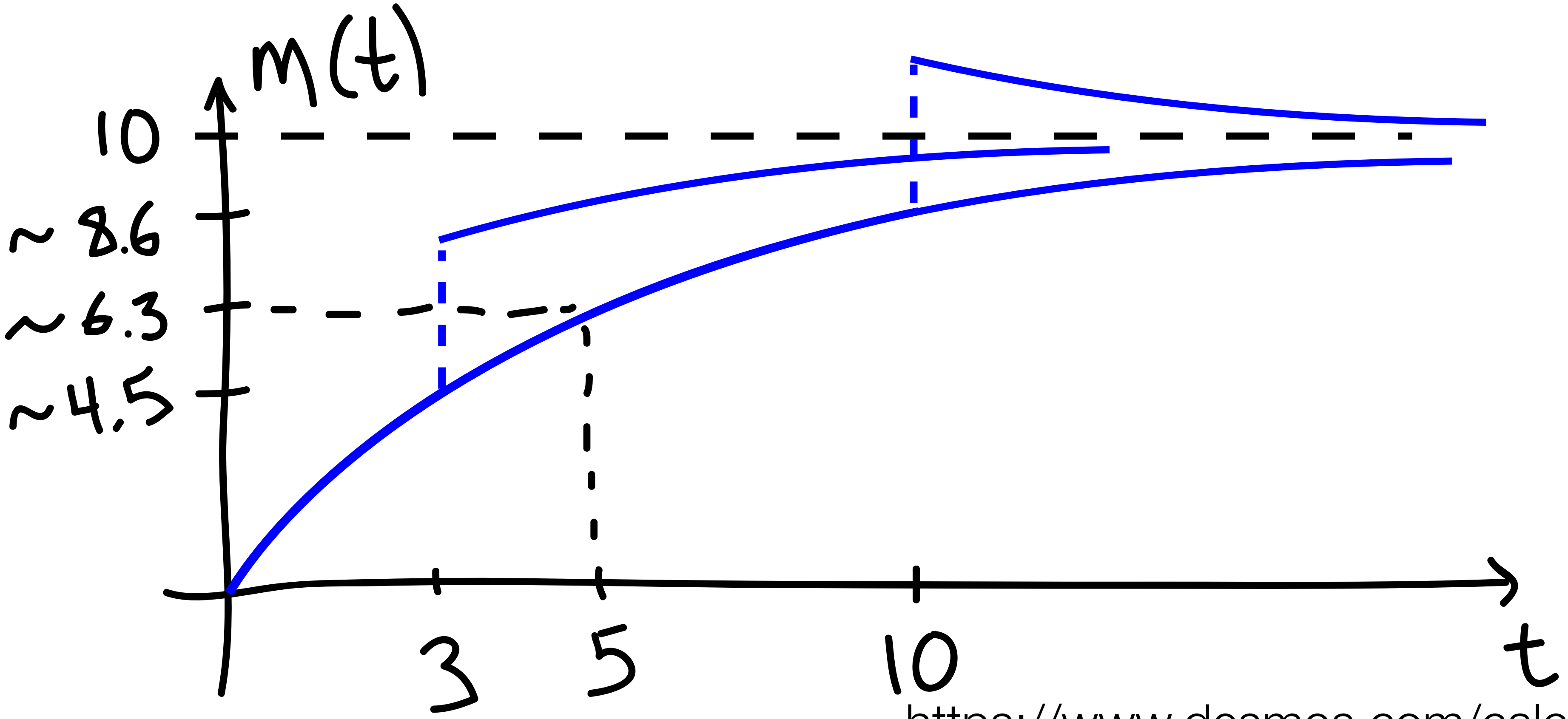
Delta-function forcing (6.5)

- Sketch the solution to the ODE.

$$m(t) = 10(1 - e^{-t/5}) + 3u_3(t)e^{-(t-3)/5}$$

$$= \begin{cases} 10(1 - e^{-t/5}) & \text{for } t < 3, \\ 10 - (10 - 3e^{3/5})e^{-t/5} & \text{for } t \geq 3. \end{cases}$$

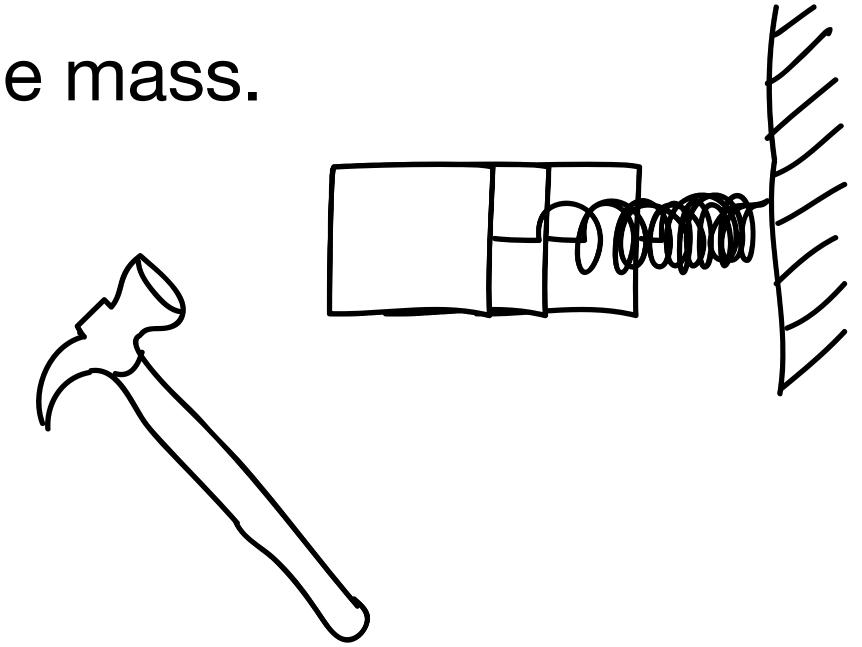
- How would it differ if t_{cube}=10 min?



https://www.desmos.com/calculator/eizjtgf3jc

Delta-function forcing (6.5)

- A hammer hits a mass-spring system imparting an impulse of $I_0 = 2 \text{ N s}$ at $t = 5 \text{ s}$. The mass of the block is $m = 1 \text{ kg}$. The drag coefficient is $\gamma = 2 \text{ kg/s}$ and the spring constant is $k = 10 \text{ kg/s}^2$. The mass is initially at $y(0) = 2 \text{ m}$ with velocity $y'(0) = 0 \text{ m/s}$.
  - Write down an equation for the position of the mass.



(A) $y'' + 2y' + 10y = 2\ u_0(t)$

(B) $y'' + 2y' + 10y = 2\ u_5(t)$

(C) $y'' + 2y' + 10y = 2\ \delta(t)$

★ (D) $y'' + 2y' + 10y = 2\ \delta(t-5)$

$$s^2Y - 2s + 2sY - 4 + 10Y = 2e^{-5c}$$

$$Y(s) = \frac{2(e^{-5s} + s + 2)}{s^2 + 2s + 10}$$

Delta-function forcing (6.5)

- Inverting Y(s)... (go through this on your own)

$$
\begin{aligned}
Y(s) &= \frac{2(e^{-5s}+s+2)}{s^2+2s+10} = \frac{2e^{-5s}}{s^2+2s+10} + 2\frac{s+2}{s^2+2s+10} \\
&= \frac{2e^{-5s}}{s^2+2s+10} + 2\frac{s+2}{(s+1)^2+9} \\
&= \frac{2e^{-5s}}{s^2+2s+10} + 2\frac{s+1}{(s+1)^2+9} + \frac{2}{(s+1)^2+9} \\
&= \frac{2e^{-5s}}{s^2+2s+10} + 2\frac{s+1}{(s+1)^2+9} + \frac{2}{3}\frac{3}{(s+1)^2+9} \\
&= \frac{2}{3}\frac{3e^{-5s}}{(s+1)^2+9} + 2\frac{s+1}{(s+1)^2+9} + \frac{2}{3}\frac{3}{(s+1)^2+9}
\end{aligned}
$$

$$
y(t) = \underbrace{\frac{2}{3}u_5(t)e^{-(t-5)}\sin(3(t-5))}_{\text{particular solution from } \delta \text{ forcing}} + \underbrace{2e^{-t}\cos(3t) + \frac{2}{3}e^{-t}\sin(3t)}_{\text{homogeneous part}}
$$

Convolution (6.6)

- We often end up with transforms to invert that are the product of two known transforms. For example,

$$Y(s) = \frac{2}{s^2(s^2+4)} = \frac{1}{s^2} \cdot \frac{2}{s^2+4}$$

- Can we express the inverse of a product in terms of the known pieces?

$$F(s)G(s) = \mathcal{L}\{??\}$$

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt \quad \rightarrow \quad F(s) = \int_0^\infty e^{-s\tau} f(\tau)\ d\tau$$

$$G(s) = \int_0^\infty e^{-st} g(t)\ dt \quad \rightarrow \quad G(s) = \int_0^\infty e^{-sw} g(w)\ dw$$

Convolution (6.6)

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau)\ d\tau \int_0^\infty e^{-sw} g(w)\ dw$$

$$= \int_0^\infty e^{-sw} g(w) \int_0^\infty e^{-s\tau} f(\tau)\ d\tau\ dw$$

$$= \int_0^\infty g(w) \int_0^\infty e^{-s(\tau+w)} f(\tau)\ d\tau\ dw$$

Replace τ using $u = \tau + w$ where w is constant in the inner integral.

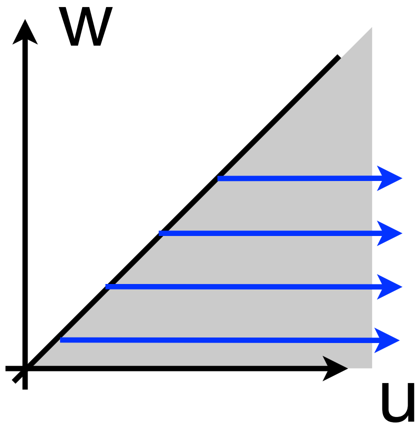
$$= \int_0^\infty g(w) \int_w^\infty e^{-s(u)} f(u-w)\ du\ dw$$

$$= \int_0^\infty \int_w^\infty e^{-su} g(w) f(u-w)\ du\ dw$$

$$= \int_a^b \int_c^d e^{-su} g(w) f(u-w)\ dw\ du$$

Convolution (6.6)

- What are the correct values for a, b, c and d?



$$\int_0^\infty \int_{u=w}^{u=\infty} e^{-su} g(w) f(u-w)\ du\ dw$$

w=constant

$$= \int_a^b \int_c^d e^{-su} g(w) f(u-w)\ dw\ du$$

(A) Integrate in u from a=0 to b=∞ and in w from c=u, d=∞.

(B) Integrate in u from a=0 to b=w and in w from c=0 to d=∞.

★ (C) Integrate in u from a=0 to b=∞ and in w from c=0 to d=u.

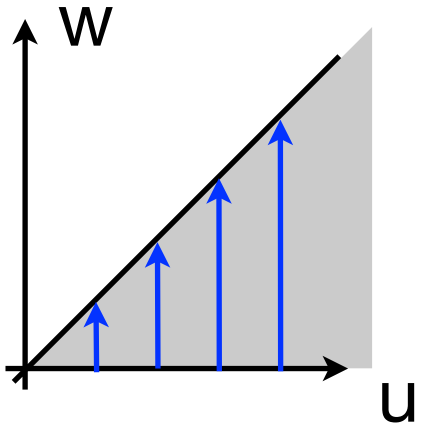
(D) Integrate in u from a=0 to b=∞ and in w from c=w to d=∞.

(E) Huh?

Convolution (6.6)

- What are the correct values for a, b, c and d?

$$\int_0^\infty \int_w^\infty e^{-su} g(w) f(u-w)\ du\ dw$$



$$= \int_a^b \int_c^d e^{-su} g(w) f(u-w)\ dw\ du$$

(A) Integrate in u from a=0 to b=∞ and in w from c=u, d=∞.

(B) Integrate in u from a=0 to b=w and in w from c=0 to d=∞.

★ (C) Integrate in u from a=0 to b=∞ and in w from c=0 to d=u.

(D) Integrate in u from a=0 to b=∞ and in w from c=w to d=∞.

(E) Huh?

Convolution (6.6)

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau)\ d\tau \int_0^\infty e^{-sw} g(w)\ dw$$

$$= \int_0^\infty \int_0^u e^{-su} g(w) f(u-w)\ dw\ du$$

$$= \int_0^\infty e^{-su} \int_0^u g(w) f(u-w)\ dw\ du$$

$$= \int_0^\infty e^{-su} h(u)\ du\ = H(s)$$

The transform of a convolution is the product of the transforms.

$$h(t) = f * g(t) = \int_0^u g(w) f(t-w)\ dw$$

$$\Rightarrow H(s) = F(s)G(s)$$

where $h(u) = \int_0^u g(w) f(u-w)\ dw$

This is called the convolution of f and g. Denoted $f * g$.

Convolution (6.6)

- To invert $Y(s) = \frac{1}{s^2} \cdot \frac{2}{s^2+4}$, we can use the fact that the inverse is the convolution of the inverses of the two pieces (instead of PFD...).

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = \; t$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \; \sin(2t)$$

$$f * g = g * f$$

$$\int_0^t f(t-w)g(w)\ dw = \int_0^t f(t)g(t-w)\ dw$$

$y(t) =$

★ (A) $\int_0^t (t-w)\sin(2w)\ dw$

★ (C) $\int_0^t w\sin(2(t-w))\ dw$

(B) $\int_0^t (t-w)\sin(2t)\ dw$

(D) $\int_0^t w\sin(2(w-t))\ dw$

Convolution (6.6)

- Transfer functions

$$ay'' + by' + cy = g(t), \quad y(0) = 0,\ y'(0) = 0$$

$$Y(s) = \frac{1}{as^2 + bs + c} G(s)$$

- Define the transfer function for the ODE:

$$H(s) = \frac{1}{as^2 + bs + c}$$

Independent of g(t)!

$$y(t) = (h * g)(t)$$

- h(t) is called the impulse response because it solves (1) when g(t)=δ(t).

$$g(t) = \delta(t)$$

$$G(s) = e^{-0s} = 1$$

$$Y(s) = \frac{1}{as^2 + bs + c}$$

$$y_{IR}(t) = h(t) = \mathcal{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\}$$

Convolution (6.6)

- Interpreting the transfer function in a model of memory.
- Your contact list got deleted. You are forced to memorize phone numbers. Let n(t) be the number of phone numbers you remember at time t. You forget numbers at a rate k. Finally, g(t) is the number of phone numbers per unit time that you memorize at time t.
- Equation: $n' = -kn + g(t)$
- Transform of n(t): $N(s) = \dfrac{G(s)}{s+k}$
- Impulse response: $H(s) = \dfrac{1}{s+k}$

$$h(t) = e^{-kt}$$

$$n(t) = \int_0^t h(t-w)g(w)\ dw\ = \int_0^t e^{-k(t-w)}g(w)\ dw$$

Convolution (6.6)

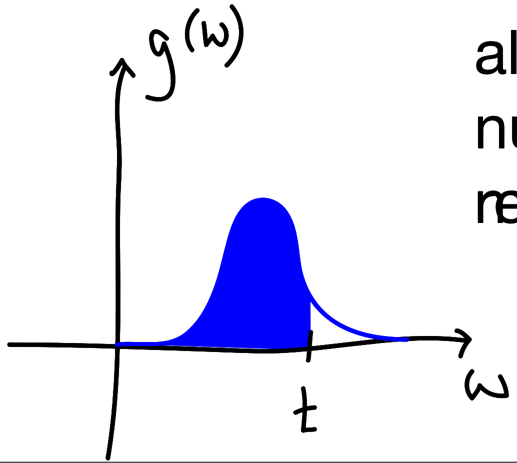
- Interpreting the transfer function in a model of memory.
- Your contact list got deleted. You are forced to memorize phone numbers. Let n(t) be the number of phone numbers you remember at time t. You forget numbers at a rate k. Finally, g(t) is the number of phone numbers per unit time that you memorize at time t.
- Equation: $n' = -kn + g(t)$
- If you memorize one phone number at t=0 (g(t)=δ(t)), h(t) tells you what's left of that memory at time t.

$$h(t) = e^{-kt}$$

- If you memorize numbers over time (some complicated g(t)),

$$n(t) = \int_0^t h(t-w)g(w)\ dw$$

$$= \int_0^t e^{-(t-w)}g(w)\ dw$$



all phone numbers ever memorized

all phone numbers still remembered

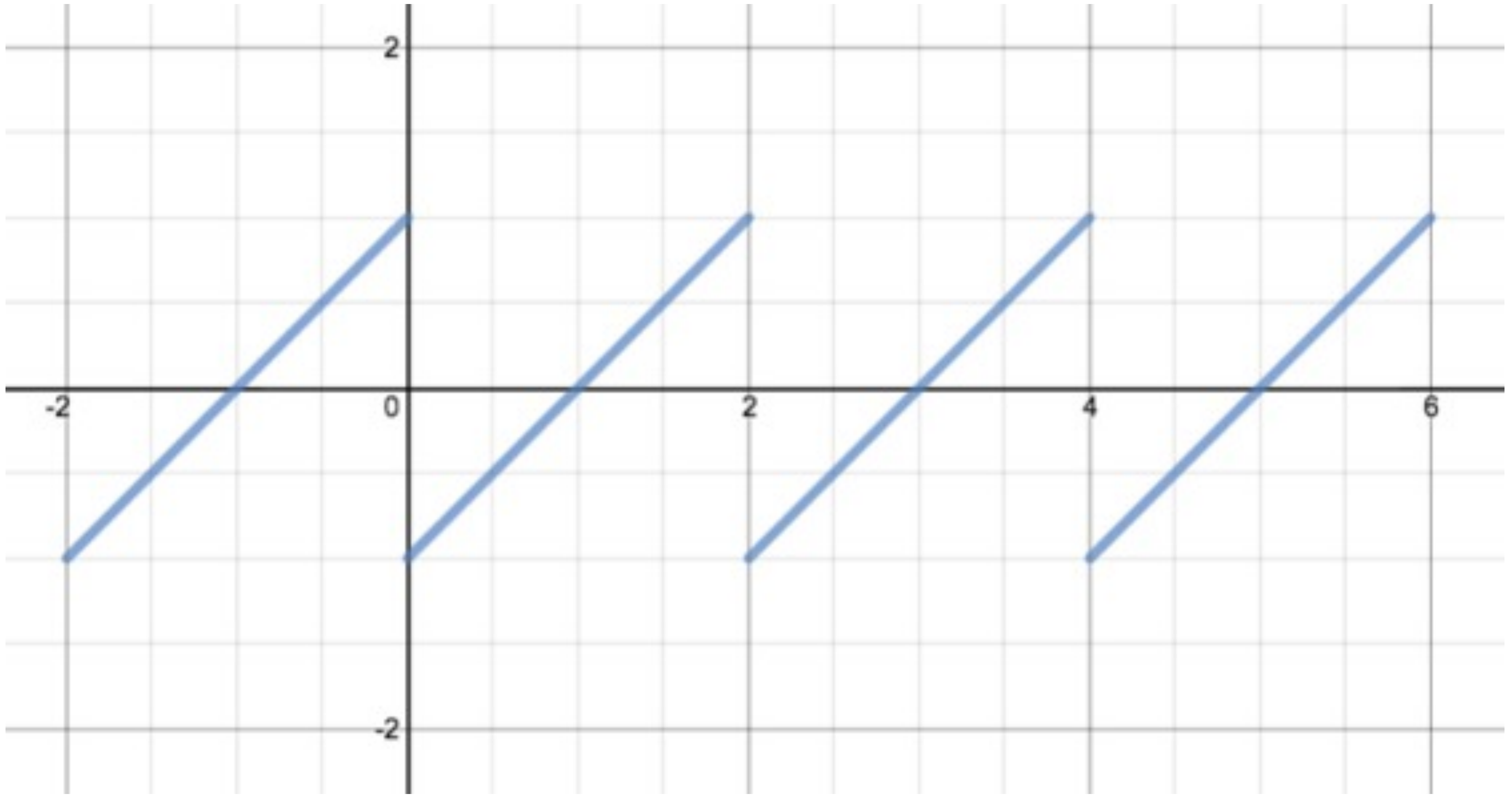
Fourier series

- Recall Method of Undetermined Coefficients for equations of the form

$$ay'' + by' + cy = f(t)$$

- Applicable for functions f(t) that are polynomials, exponentials, sin, cos and products of those.

- How about functions like this (period but not trig)?



- What if we could construct such functions using only sine and cosine functions?