MATH 256 – Midterm 2 – March 15, 2016.

Last name: ______________ First name: ______________ Student #: ______________

Tutorial (circle one): T2A - Cole, T2B - Shirin, T2C - Dhananjay, T2D - Xiaoyu, T2E - Catherine, T2F - Will

Place a box around each answer so that it is clearly identified. Point values are approximate and may differ slightly in the final marking scheme.

1. **[6 pts]** Consider the system of equations $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{pmatrix} a & a-3 \\ a-3 & a \end{pmatrix}.$$

(a) In each row of the table below, give inequalities/equations involving a which ensure that the steady state is of the given type. Write X if there is no value of a that gives the specified classificaiton.

1

$\lambda^2 - 2a\lambda + a^2 - (a-3)^2 = 0$

$\lambda^2 - 2a\lambda + 6a - 9 = 0$

$\lambda = \frac{2a \pm \sqrt{4a^2 - 24a + 36}}{2}$

$= a \pm \sqrt{a^2 - 6a + 9}$

$= a \pm (a-3)$

1 $\lambda_1 = 3, \quad \lambda_2 = 2a - 3$

If table all wrong, part marks

Type	Condtion(s) on a
unstable node	$a > 3/2$
unstable spiral	X
stable node	X
stable spiral	X
saddle	$a < 3/2$
repeated eigenvalue	$a = 3$
zero eigenvalue	$a = 3/2$

(-1) for any incorrect

2. **[5 pts]** Find the general solution to the system of equations $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

→ Remembering formula ok for last three points (-1 for simple error)

$\lambda^2 - 4\lambda + 5 = 0$

$\lambda^2 - 4\lambda + 4 + 1 = 0$

$(\lambda-2)^2 + 1 =$

$\lambda = 2 \pm i$ 1

$\lambda = 2+i$: $\begin{pmatrix} 2-2-i & -1 \\ 1 & 2-2-i \end{pmatrix} = \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}$

(Check $\det(A - \lambda I) = i^2 - 1 = 0$!)

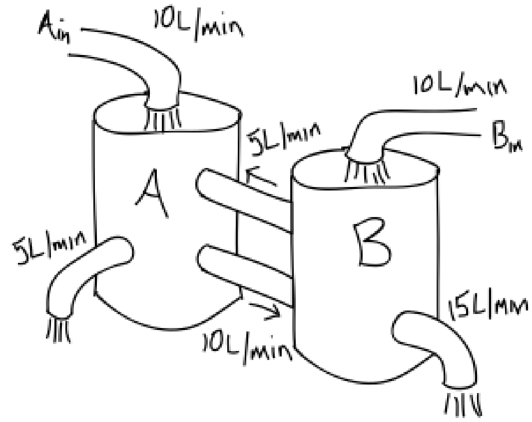
$\vec{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ 1

$e^{(2+i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} = e^{2t}(\cos t + i\sin t)\begin{pmatrix} i \\ 1 \end{pmatrix} = e^{2t}\begin{pmatrix} i\cos t - \sin t \\ \cos t + i\sin t \end{pmatrix}$

$= e^{2t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t + i\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t\right)\right)$ 1 1

1 $\boxed{x(t) = e^{2t}\left[A\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t\right] + B\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t\right]\right]}$

3. **[7 pts]** Suppose tank A and B each hold 5 L. The upper pipe dumping solution into tank A (marked A_{in}) has a salt concentration of 2 g/L and the one dumping solution in tank B (marked B_{in}) has a salt concentration of 3 g/L.



(a) Write down a system of equations for the problem.

1 for overall form

$$A' = 20 - \frac{15A}{5} + \frac{5B}{5}$$

$$B' = 30 + \frac{10A}{5} - \frac{20B}{5}$$

1 1 1 1

(b) Find the steady state of the system.

$$\begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -20 \\ -30 \end{pmatrix}$$ 1

$$\left(\begin{array}{cc|c} -3 & 1 & -20 \\ 2 & -4 & -30 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 3 & 50 \\ 2 & -4 & -30 \end{array}\right)$$

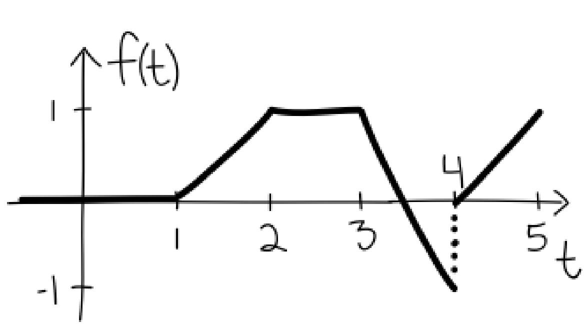
$$\left(\begin{array}{cc|c} 1 & 3 & 50 \\ 0 & -10 & -130 \end{array}\right)$$

$B = 13$

$A = 50 - 3 \cdot 13 = 11$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$ 1

4. **[6 pts]** Write an expression for the function $f(t)$ shown below using Heaviside functions. In your final answer, all terms should be in the form $u_c(t)g(t-c)$ for some g, such that the Laplace transform is easy to compute.



$$f(t) = u_1(t)(t-1) - u_2(t)(t-2) - 2u_3(t)(t-3) + 3u_4(t)(t-4) + u_4(t).$$

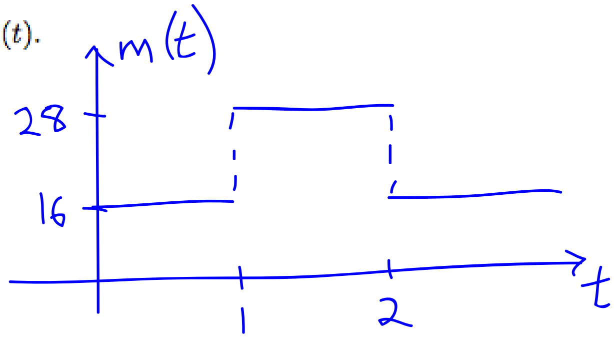
(-1) for each incorrect term

(-2) for not putting into $u_c(t)g(t-c)$ (-1 if only one term)

(-1) for writing $u_c(t-c)$ instead of $u_c(t)\cdot(t-c)$

5. **[15 pts]** Consider the function $m(t) = 16 + 12(u_1(t) - u_2(t))$.

(a) Sketch $m(t)$.



-1 for each missing jump

-1 for each wrong constant level

Worth 3 pts total.

(b) Solve the differential equation $y' + 4y = m(t) + 3\delta(t-3)$ with initial condition $y(0) = 0$.

$$sY(s) - \underset{1}{0} + 4Y(s) = \underset{1}{\frac{16}{s}} + 12\,\underset{1}{\frac{e^{-s} - e^{-2s}}{s}} + \underset{1}{3e^{-3s}}$$

$$Y(s) = \frac{16}{s(s+4)} + \frac{12}{s(s+4)}\left(e^{-s} - e^{-2s}\right) + \frac{3}{s+4}e^{-3s}$$

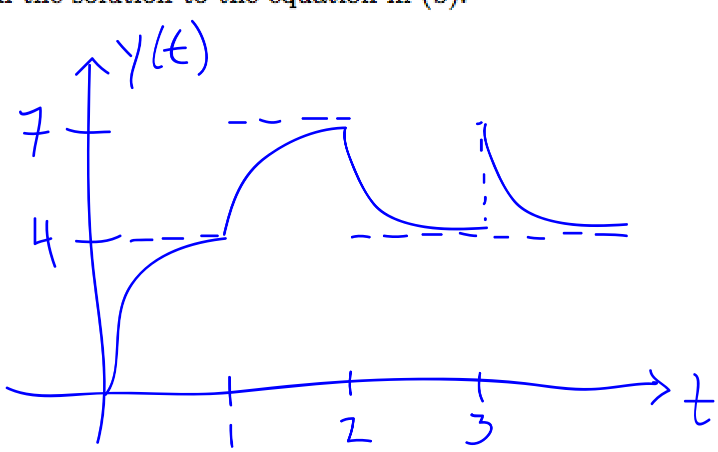
2 for PFD

$$= \frac{4}{s} - \frac{4}{s+4} + \left(\frac{3}{s} - \frac{3}{s+4}\right)\left(e^{-s} - e^{-2s}\right) + \frac{3}{s+4}e^{-3s}$$

$$y(t) = \underset{1}{4\left(1 - e^{-4t}\right)} + \underbrace{3u_1(t)\left(1 - e^{-4(t-1)}\right) - 3u_2(t)\left(1 - e^{-4(t-2)}\right)}_{1}$$

$$+ \underset{1}{3u_3(t)e^{-4(t-3)}}$$

(c) Sketch the solution to the equation in (b).



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6. **[5 pts]** The motion of a forced tuning fork satisfies the equation $y'' + 2y' + 9y = 5\cos(\omega t)$. The Method of Undetermined Coefficients gives a particular solution of

$$y_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

where

$$A = \frac{5(9-\omega^2)}{4\omega^2 + (9-\omega^2)^2} \quad \text{and} \quad B = \frac{10\omega}{4\omega^2 + (9-\omega^2)^2}.$$

At what frequency ω does the tuning fork vibrate with largest amplitude? Using a formula for the amplitude of y_p is acceptable. Any claims (e.g. "$\omega = 42$ is a max.") must be justified.

$$\text{Amp} = \sqrt{A^2 + B^2} \quad 1$$

$$= \sqrt{\frac{25(9-\omega^2)^2 + 100\omega^2}{(4\omega^2 + (9-\omega^2)^2)^2}}$$

$$= 5\sqrt{\frac{\cancel{(9-\omega^2)^2 + 4\omega^2}}{(4\omega^2 + (9-\omega^2)^2)^{\cancel{2}}}}$$

$$= \frac{5}{\sqrt{4\omega^2 + (9-\omega^2)^2}} \quad 1$$

$p(\omega) = 4\omega^2 + (9-\omega^2)^2$ 1 has min so Amp has max:

$p'(\omega) = 8\omega + 2(9-\omega^2)(-2\omega)$

$= 4\omega^3 - 28\omega = 0$ when $\omega = 0$ or $\omega = \pm\sqrt{7}$ 1

$p''(\omega) = 12\omega^2 - 28$

$p''(0) = -28 < 0$ so max of $p(\omega)$

$p''(\pm\sqrt{7}) = 12 \cdot 49 - 28 > 0$ so min of $p(\omega)$ 1

Amp has max at $\omega = \pm\sqrt{7}$.

Hint: Recall that taking the square root of a function does not change the location of its minima and the reciprocal of a function has maxima wherever the original function has minima.

7. (a) **[3 pts]** Calculate the inverse transform of $Y(s) = \dfrac{s}{s^2 + 8s + 20}$.

$$Y(s) = \frac{s}{(s+4)^2 + 4} = \frac{s+4}{(s+4)^2 + 2^2} - 2\,\frac{2}{(s+4)^2 + 2^2}$$

1 1

$$y(t) = e^{-4t}\left(\cos 2t - 2\sin 2t\right)$$

1

(b) **[3 pts]** Give a differential equation and initial values that would have a transformed solution $Y(s)$.

$$\boxed{y'' + 8y' + 20y = 0}$$

1

$$s^2Y - s\,y(0) - y'(0) + 8sY - 8y(0) + 20Y = 0$$

$$\boxed{y(0) = 1} \quad \boxed{y'(0) = -8}$$

1 1

8. **[5 pts]** Consider the vector field in the figure below. The absolute values of the eigenvalues are 1 and 3. The eigenvectors lie along the straight lines. Give an expression for the general solution to the differential equation associated with the vector field. You will have to determine the sign of each eigenvalue and the eigenvector associated with each eigenvalue. The sizes of the vectors are to scale.

1 $\lambda_1 = -1$ $\quad v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ 1

1 $\lambda_2 = -3$ $\quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 1

$$x(t) = c_1 e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

1