Today

- Office hour Jan 17 cancelled - Jan 18 instead, 12-1pm
- Saltwater inflow example
- General solutions, independence of functions and the Wronskian
- Distinct roots of the characteristic equation
- Review of complex numbers
- Complex roots of the characteristic equation

Modeling - Example

- Saltwater with a concentration of 200 g/L flows into a tank at a rate 2 L/min. The tank starts with no salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank?

(a) The IVP is

(A) $m' = 200 - 2m, \quad m(0) = 0$

(B) $m' = 400 - 2m, \quad m(0) = 200$

★ (C) $m' = 400 - m/5, \quad m(0) = 0$

(D) $m' = 200 - m/5, \quad m(0) = 0$

(E) $m' = 400 - m/5, \quad m(0) = 200$

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(b) Directly from the equation ($m' = 400 - m/5$), find an m for which $m'=0$.

- $m=2000$. Called steady state - a constant solution.
- What happens when $m < 2000$? ---> $m' > 0$.
- What happens when $m > 2000$? ---> $m' < 0$.
- Limiting mass: 2000 g (Long way: solve the eq. and let $t \to \infty$.)

Existence and uniqueness

Theorem 2.4.2 Let the functions f and $\dfrac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta, \quad \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t_0 < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the IVP

$$y' = f(t, y), \quad y(t_0) = y_0.$$

- A couple questions/examples to explore on your own:
  - Why don't we get a solution all the way to the ends of the t interval?
    - Example: $$\frac{dy}{dt} = y^2, \quad y(0) = 1$$
  - How does a non-continuous RHS lead to more than one solution?
    - Example: $$\frac{dy}{dt} = \sqrt{y}, \quad y(0) = 0$$

Second order linear equations

- The general form for a second order linear equation:

$$y'' + p(t)y' + q(t)y = g(t)$$

- Now, an IVP requires two ICs:

$$y(0) = y_0, \quad y'(0) = v_0$$

- As with first order linear equations, we have homogeneous (g=0) and non-homogeneous second order linear equations.

- We'll start by considering the homogeneous case with constant coefficients:

$$ay'' + by' + cy = 0$$

Homogeneous equations with constant coefficients

$$ay'' + by' + cy = 0$$

- Suppose you already found a couple solutions, $y_1(t)$ and $y_2(t)$. This means that

$$ay_1'' + by_1' + cy_1 = 0 \qquad \text{and} \qquad ay_2'' + by_2' + cy_2 = 0$$

- Notice that $y(t) = C_1 y_1(t)$ is also a solution. Plug it in and check:

$$\begin{aligned} &a(C_1 y_1)'' + b(C_1 y_1)' + c(C_1 y_1) \\ &\quad = aC_1(y_1)'' + bC_1(y_1)' + cC_1(y_1) \\ &\quad = C_1(ay_1'' + by_1' + cy_1) = 0 \end{aligned}$$

Homogeneous equations with constant coefficients

- Which of the following functions are also solutions?

  (A) $y(t) = y_1(t)^2$

  ☆ (B) $y(t) = y_1(t)+y_2(t)$

  (C) $y(t) = y_1(t)\ y_2(t)$

  (D) $y(t) = y_1(t)\ /\ y_2(t)$

- In fact, the following are all solutions: $C_1y_1(t)$, $C_2y_2(t)$, $C_1y_1(t)+C_2y_2(t)$.
- With first order equations, the arbitrary constant appeared through an integration step in our methods. With second order equations, not always.
- Instead, find two independent solutions, $y_1(t)$, $y_2(t)$, by whatever method.
- The general solution will be $y(t) = C_1y_1(t) + C_2y_2(t)$.

Homogeneous equations with constant coefficients

- One case where the arbitrary constants DO appear as we calculate:

$$y'' + y' = 0$$

$$y' + y = C_1$$

$$e^t y' + e^t y = C_1 e^t$$

$$(e^t y)' = C_1 e^t$$

$$e^t y = C_1 e^t + C_2$$

$$y = C_1 + C_2 e^{-t}$$

- More common would be that we find solutions y(t) = 1 and y(t)= e^{-t} and simply write down

$$y = C_1 + C_2 e^{-t}$$

Homogeneous equations with constant coefficients

- So in general how do we find the two independent solutions y_1 and y_2?
- Exponential solutions seem to be common so let's assume $y(t)=e^{rt}$ and see if that gets us anything useful..
- Solve $y'' + y' = 0$ by assuming $y(t) = e^{rt}$ for some constant r.

$$(e^{rt})'' + (e^{rt})' = 0$$

$$r^2 e^{rt} + r e^{rt} = 0$$

$$r^2 + r = 0$$

$$r(r+1) = 0$$

$$r = 0, \quad r = -1$$

$$y = C_1 e^0 + C_2 e^{-t}$$

$$y = C_1 + C_2 e^{-t}$$

Homogeneous equations with constant coefficients

- Solve $y'' - 4y = 0$ subject to the ICs $y(0) = 3, y'(0) = 2$.

(A) $y(t) = C_1 e^{2t} + C_2 e^{-2t}$

☆ (B) $y(t) = 2e^{2t} + e^{-2t}$

(C) $y(t) = \frac{7}{4}e^{4t} + \frac{5}{4}e^{-4t}$

(D) $y(t) = e^{2t} + 2e^{-2t}$

(E) $y(t) = C_1 e^{4t} + C_2 e^{-4t}$

$$(e^{rt})'' - 4(e^{rt}) = 0$$

$$r^2(e^{rt}) - 4(e^{rt}) = 0$$

$$r^2 - 4 = 0$$

$$r = 2, -2$$

$$y(t) = C_1 e^{2t} + C_2 e^{-2t}$$

$$y(0) = C_1 + C_2 = 3$$

$$y'(0) = 2C_1 - 2C_2 = 2$$

Homogeneous equations with constant coefficients

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- There are three cases.
    - Two distinct real roots: $b^2 - 4ac > 0$. ($r_1 \neq r_2$)
    - A repeated real root: $b^2 - 4ac = 0$.
    - Two complex roots: $b^2 - 4ac < 0$.
- For case i, we get $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.
- Do our two solutions cover all possible ICs? That is, can we use them to form a general solution?

Independence and the Wronskian (Section 3.2)

- Example: Suppose $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are two solutions to some equation. Can we solve ANY initial condition $y(0) = y_0, \; y'(0) = v_0$ with these two solutions?

$$y(t) = C_1 e^{2t+3} + C_2 e^{2t-3}$$

$$y(0) = C_1 e^{3} + C_2 e^{-3} = y_0$$

$$y'(0) = 2C_1 e^{3} + 2C_2 e^{-3} = v_0$$

- Solve this system for C_1, C_2...

- Can't do it. Why? $\begin{pmatrix} e^3 & e^{-3} \\ 2e^3 & 2e^{-3} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$

$$\det \begin{pmatrix} e^3 & e^{-3} \\ 2e^3 & 2e^{-3} \end{pmatrix} = 0$$

Independence and the Wronskian (Section 3.2)

- For any two solutions to some linear ODE, to ensure that we have a general solution, we need to check that

$$\det \begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = y_1(0)y_2'(0) - y_1'(0)y_2(0) \neq 0$$

- For ICs other than t_0=0, we require that

$$y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$$

- This quantity is called the Wronskian.

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

Independence and the Wronskian (Section 3.2)

- Two functions $y_1(t)$ and $y_2(t)$ are linearly independent provided that the only way that $C_1y_1(t) + C_2y_2(t) = 0$ for all values of t is when $C_1=C_2=0$.

  e.g. $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are not independent.

  Find values of $C_1\neq0$ and $C_2\neq0$ so that $C_1y_1(t) + C_2y_2(t) = 0$.

(A) $C_1 = e^{-2t-3},\ C_2 = -e^{-2t+3}$

(B) $C_1 = e^{-2t+3},\ C_2 = -e^{-2t-3}$

(C) $C_1 = e^{-3},\ C_2 = e^{3}$

★ (D) $C_1 = e^{-3},\ C_2 = -e^{3}$

(E) $C_1 = e^{3},\ C_2 = -e^{-3}$

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- The Wronskian is defined for any two functions, even if they aren't solutions to an ODE.

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

- If the Wronskian is nonzero for some t, the functions are linearly independent.

- If $y_1(t)$ and $y_2(t)$ are solutions to an ODE and the Wronskian is nonzero then they are independent and

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is the general solution. We call $y_1(t)$ and $y_2(t)$ a fundamental set of solutions and we can use them to solve any IC.

Independence and the Wronskian (Section 3.2)

- So for case i (distinct roots), can we form a general solution from

$$y_1(t) = e^{r_1 t} \text{ and } y_2(t) = e^{r_2 t}\text{?}$$

- Must check the Wronskian:

$$W(e^{r_1 t}, e^{r_2 t})(t) = e^{r_1 t} r_2 e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t}$$

$$= (r_1 - r_2) e^{r_1 t} e^{r_2 t} \neq 0$$

So yes! $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is the general solution.