Today

- Fourier series for Method of Undetermined Coefficients
- Fourier series for Heat / Diffusion equation

Fourier series (Method Undetermined Coefficients)

- Replace f(t) by a sum of trig functions, if possible:

$$ay'' + by' + cy = f(t) \stackrel{?}{=} A_0 + \sum_{n=1}^{N} a_n \cos(\omega_n t) + \sum_{n=1}^{N} b_n \sin(\omega_n t)$$

- For any f(t), how do we find the best choice of A_0, a_n, b_n?

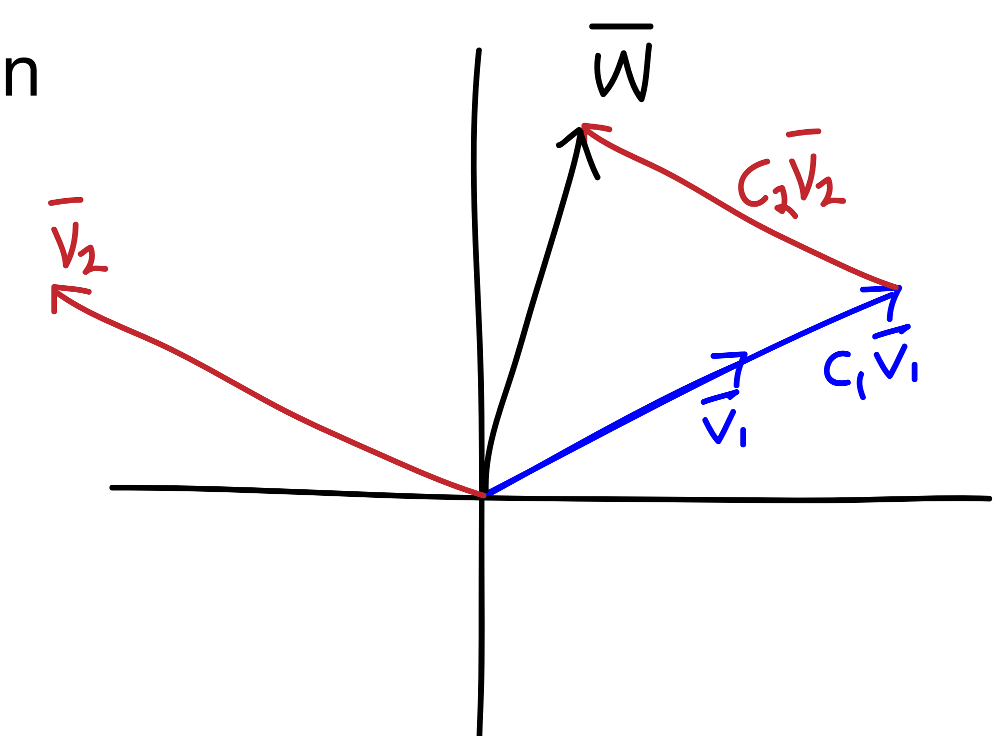
- This problem is closely related to an analogous vector problem: how do you choose c_1, c_2 so that $w = c_1 v_1 + c_2 v_2$?

- If v_1 and v_2 are perpendicular ($v_1 \circ v_2 = 0$), then

$$\mathbf{w} \circ \mathbf{v_1} = c_1 \mathbf{v_1} \circ \mathbf{v_1} + \cancel{c_2 \mathbf{v_2} \circ \mathbf{v_1}}$$

$$c_1 = \frac{\mathbf{w} \circ \mathbf{v_1}}{\mathbf{v_1} \circ \mathbf{v_1}}$$

$$\mathbf{v_1} \circ \mathbf{v_1} = ||\mathbf{v_1}||^2 \qquad c_2 = \frac{\mathbf{w} \circ \mathbf{v_2}}{\mathbf{v_2} \circ \mathbf{v_2}}$$



Fourier series (Method Undetermined Coefficients)

- For functions, define dot product as

$$g(t) \circ h(t) = \int_{\text{one period}} g(t)h(t)\ dt$$

- just like for vectors but indexed over all t instead of 1, 2, 3:

$$\mathbf{v} \circ \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Fourier series (Method Undetermined Coefficients)

- Back to our ODE, what do we choose for the w_n if f(t) has period T? Keep in mind that we want all the functions involved to have period T.

$$ay'' + by' + cy = f(t) \stackrel{?}{=} A_0 + \sum_{n=1}^{N} a_n \cos(\omega_n t) + \sum_{n=1}^{N} b_n \sin(\omega_n t)$$

Once we find the coefficients, this will be the Fourier series representation of f(t).

(A) $w_n = \pi / T$

(B) $w_n = 2 \pi / T$

(C) $w_n = n \pi / T$

(D) $w_n = 2 \pi n / T$

(E) Don't know. Explain please.

Draw graphs on doc cam.

Fourier series (Heat/Diffusion equation)

- When we talk about the Heat/Diffusion equation, we'll need to satisfy conditions at x=0 and x=L (ends of a heated rod or a pipe filled with solution):

$$u(0) = 0, \; u(L) = 0$$

- How should we choose w_n in this case?

$$u(x) = A_0 + \sum_{n=1}^{N} a_n \cos(\omega_n x) + \sum_{n=1}^{N} b_n \sin(\omega_n x)$$

(A) $w_n = \pi / L$

(B) $w_n = 2 \pi / L$

(C) $w_n = n \pi / L$

(D) $w_n = 2 \pi n / L$

(E) Don't know. Explain please.

- Here, the function is not periodic on [0,L] but rather [-L,L]!!

Fourier series (Heat/Diffusion equation)

- Want to find Fourier series coefficients A_0, a_n, b_n, that make

$$u(x) \approx A_0 + \sum_{n=1}^{N} a_n \cos(\omega_n x) + \sum_{n=1}^{N} b_n \sin(\omega_n x)$$

- This will require taking integrals (dot products) like

$$g(x) \circ h(x) = \int_{-L}^{L} g(x)h(x)\ dx$$

- This integral is zero when

(A) g is even, h is odd. <-- g(x)h(x) is odd.

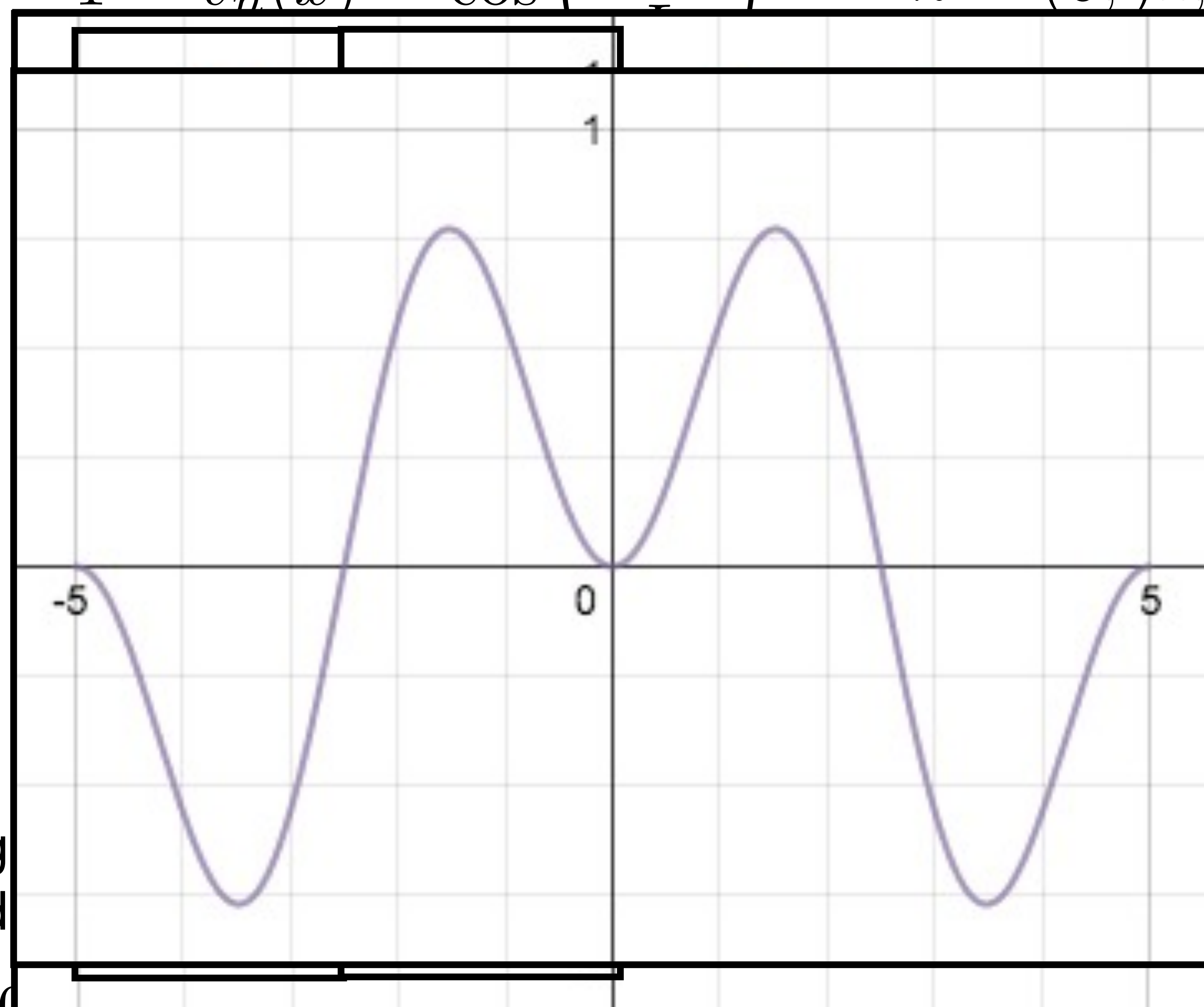
(B) g is even, h is even. <-- g(x)h(x) is even.

(C) g is odd, h is odd. <-- g(x)h(x) is even.

Fourier series (Heat/Diffusion equation)

- Define $v_0(x) = 1$ $v_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ $n = (0,)1, 2, 3, \ldots$

$\ldots$



$v_0 \circ v_n =$

$(m \neq n)$

★(A) 0

(B) π

(C) π/2

(D) nπ/2

2

/2

Integral of a trig
over one period

$v_0 \circ w_n = 0$

$$v_n \circ v_n = \int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right)\, dx = L$$

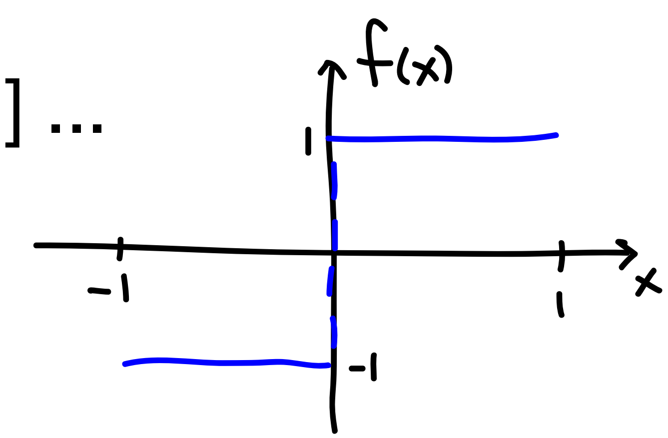
$$g(x) \circ h(x) = \int_{-L}^{L} g(x)h(x)\, dx$$

Fourier series (Heat/Diffusion equation)

- Defining Fourier series:
- Define a function $f_{FS}(x)$ on the interval [-L,L] by choosing coefficients A_0, a_n and b_n and setting

$$f_{FS}(x) = A_0 + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$
$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

- This is called a Fourier series. It may or may not converge for different values of x, depending on the choice of coefficients.
- Given any function f(x) on [-L,L], can it be represented by some $f_{FS}(x)$?
- Let's check for $f(x) = 2u_0(x)-1$ on the interval [-1,1] ...

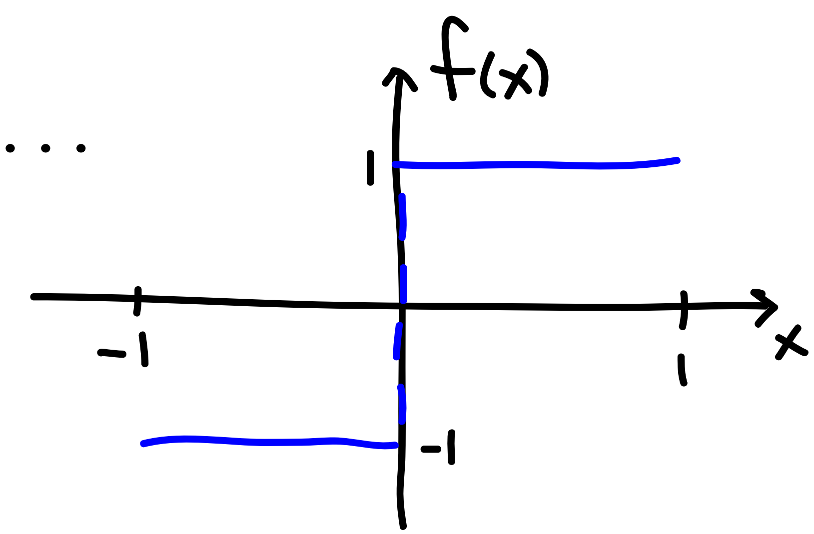


Fourier series (Heat/Diffusion equation)

- Find the Fourier series for f(x) = $2u_0(x)-1$ on the interval [-1,1].

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$

$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$



- Our hope is that f(x) = $f_{FS}(x)$ so we calculate coefficients as if they were equal:

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\ dx$$

A_0 is the average value of f(x)!

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\ dx$$

- To simplify formulas, usually define

$$a_0 = 2A_0 = \frac{1}{L} \int_{-L}^{L} f(x)\ dx$$