Today

- Fourier series calculations

Fourier series

- Calculate the coefficients of the Fourier series of a function:

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$
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$$v_0(x) = 1$$
$$v_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$
$$w_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$f_{FS}(x) = \frac{a_0}{2} v_0(x) + a_1 v_1(x) + a_2 v_2(x) + \cdots$$
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$$f_{FS}(x) \circ v_n(x) =$$

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$$f_{FS}(x) \circ v_n(x) = \cancelto{0}{\frac{a_0}{2} v_0(x) \circ v_n(x)} + \cancelto{0}{a_1 v_1(x) \circ v_n(x)} + \cancelto{0}{a_2 v_2(x) \circ v_n(x)} + \cdots$$
$$+ \cancelto{0}{b_1 w_1(x) \circ v_n(x)} + \cancelto{0}{b_2 w_2(x) \circ v_n(x)} + \cdots$$

$$= a_n v_n(x) \circ v_n(x) = a_n L$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f_{FS}(x) \cos\left(\frac{n\pi x}{L}\right) \, dx$$

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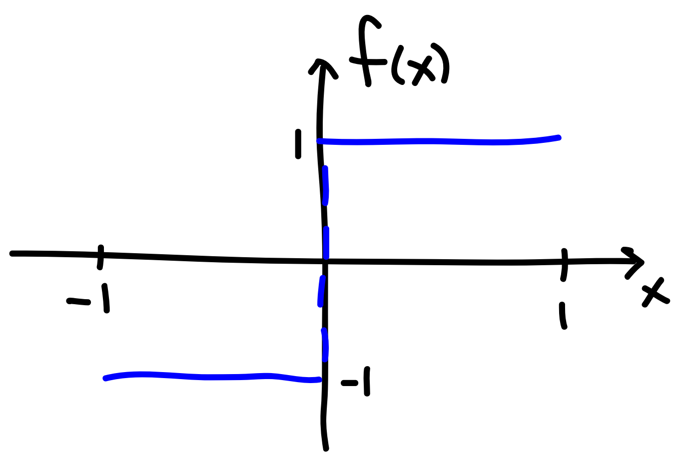

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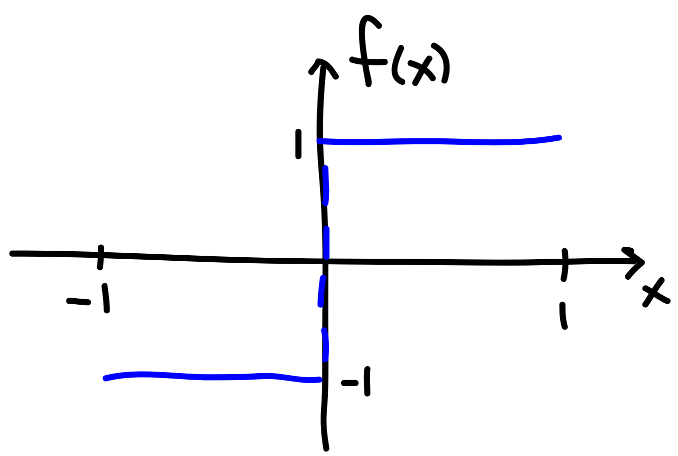

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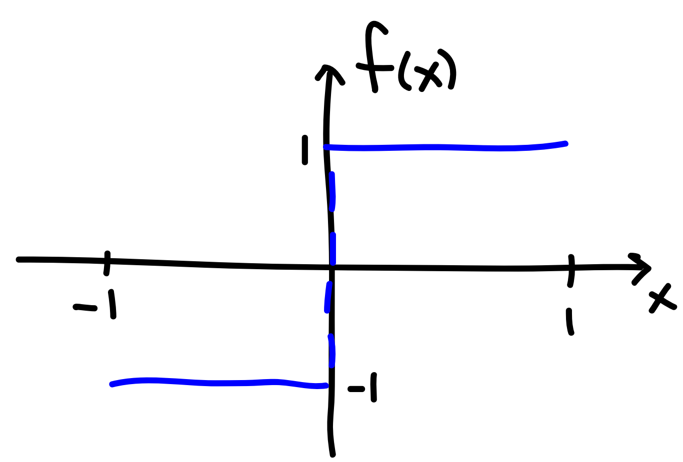
(A) 0

(B) $\frac{1}{\pi}$

(C) undefined

(D) $\frac{1}{n\pi}$



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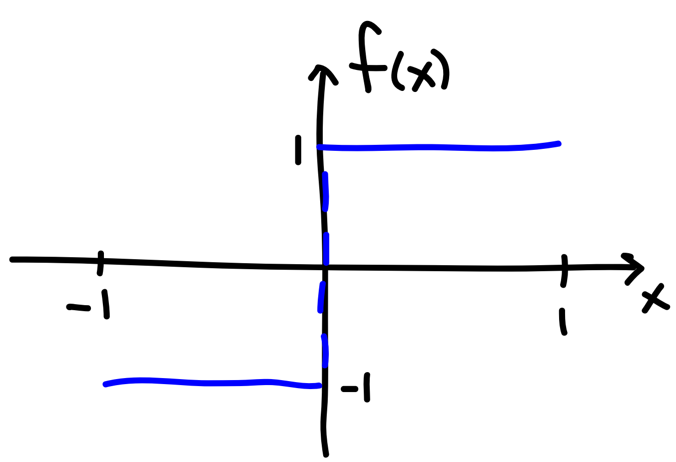

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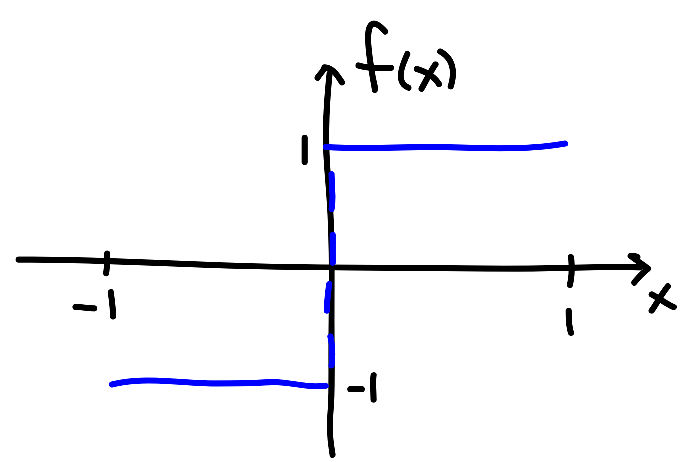

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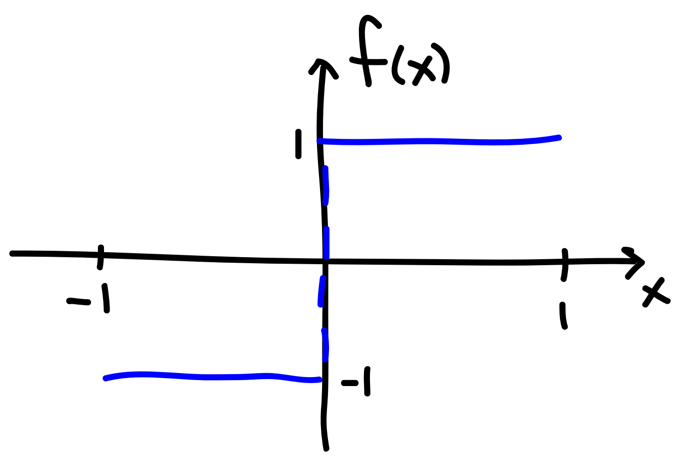

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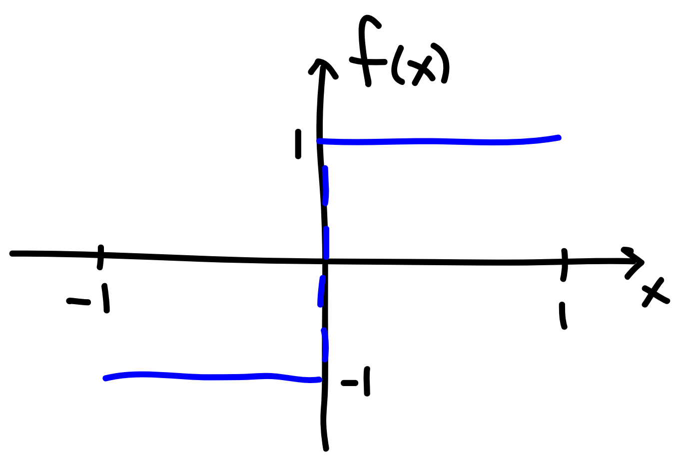

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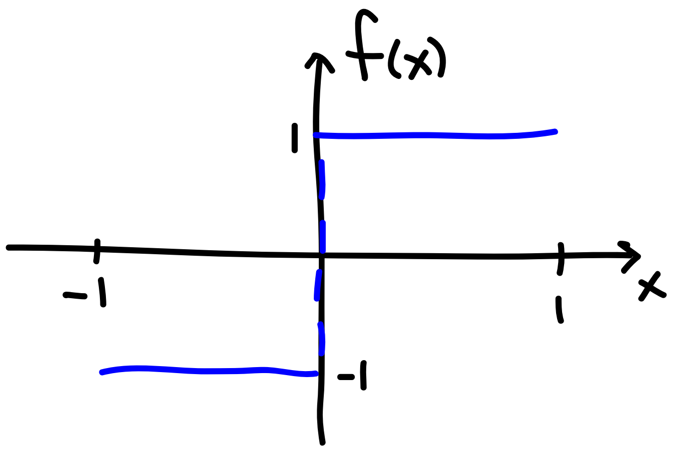

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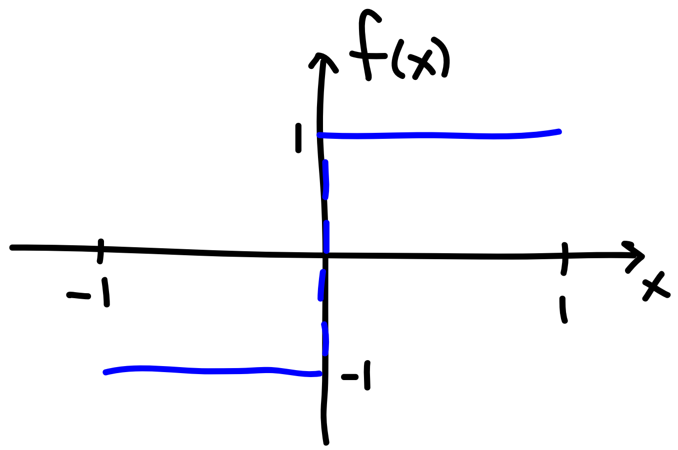

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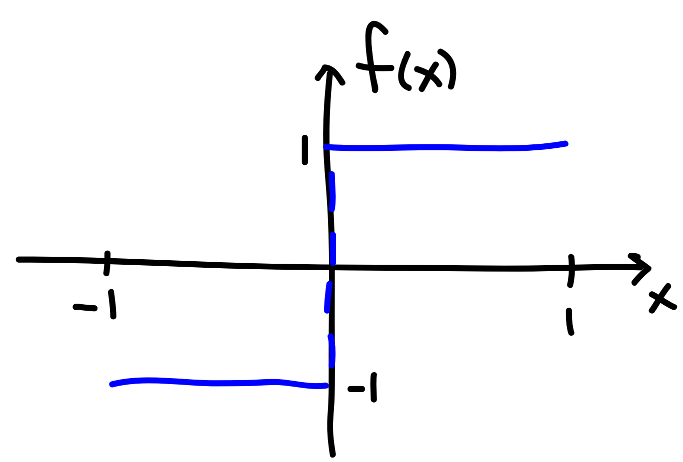

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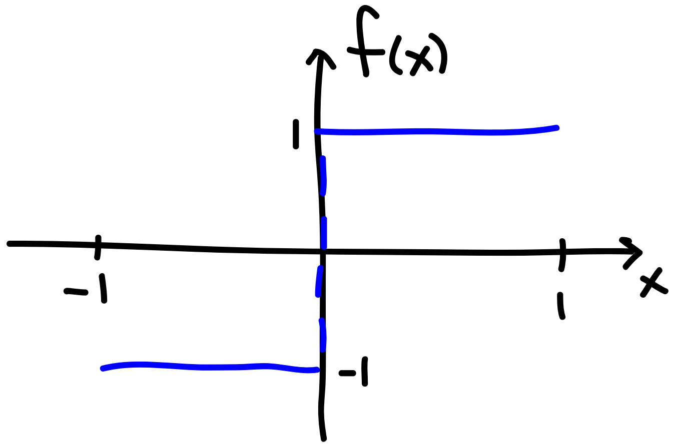

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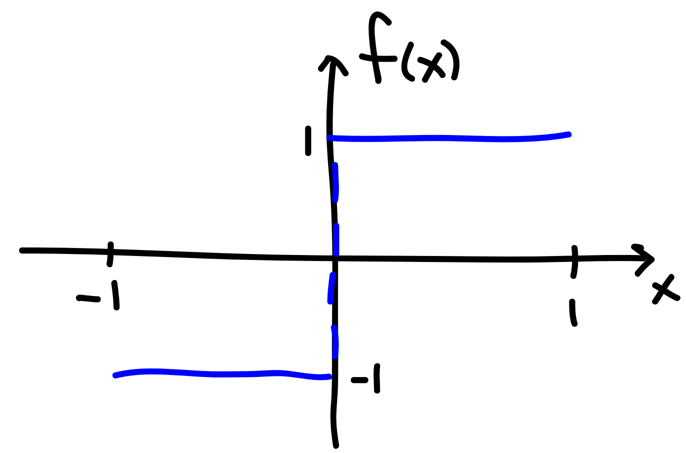

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https://www.desmos.com/calculator/tlvtikmi0y

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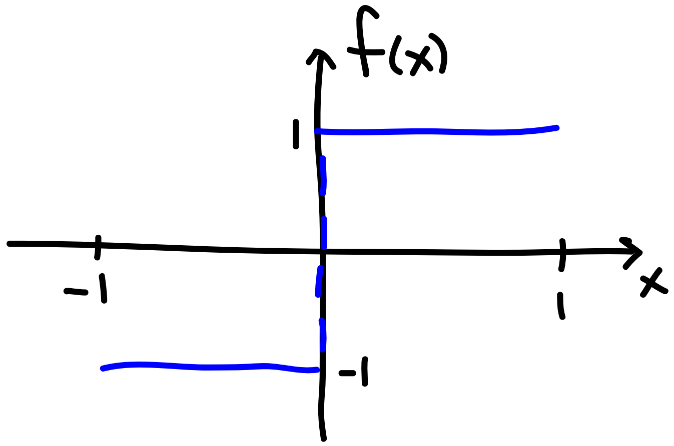

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Does f(x) = f_{FS}(x) for all x?

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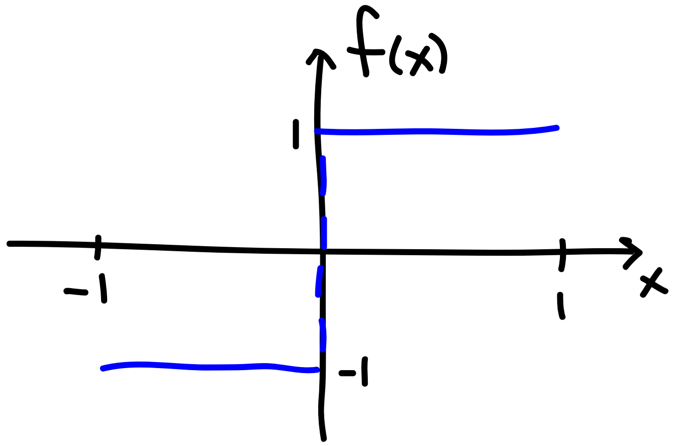

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Problems at jumps! x=-1, 0, 1

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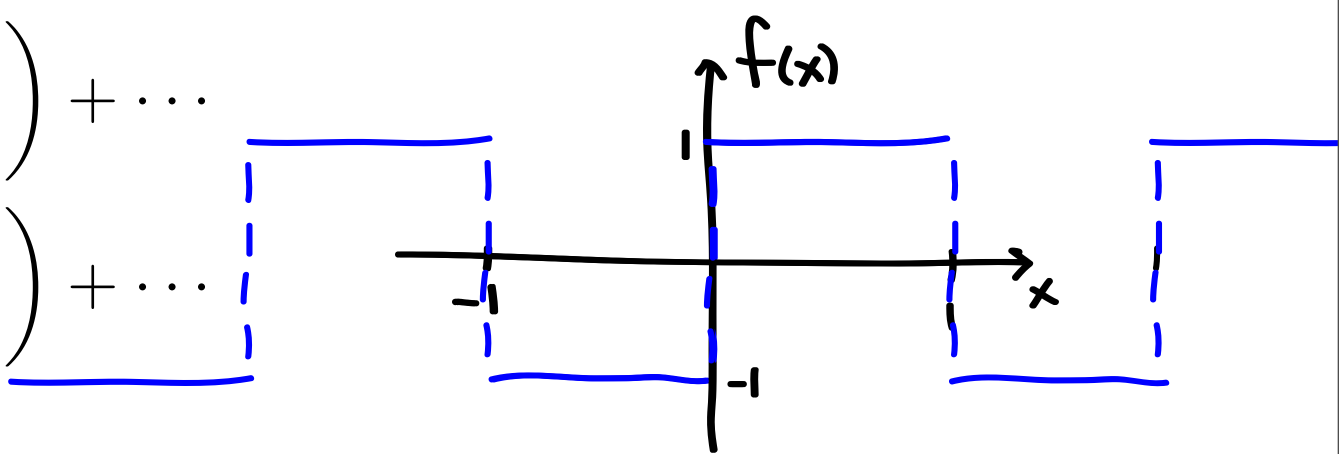

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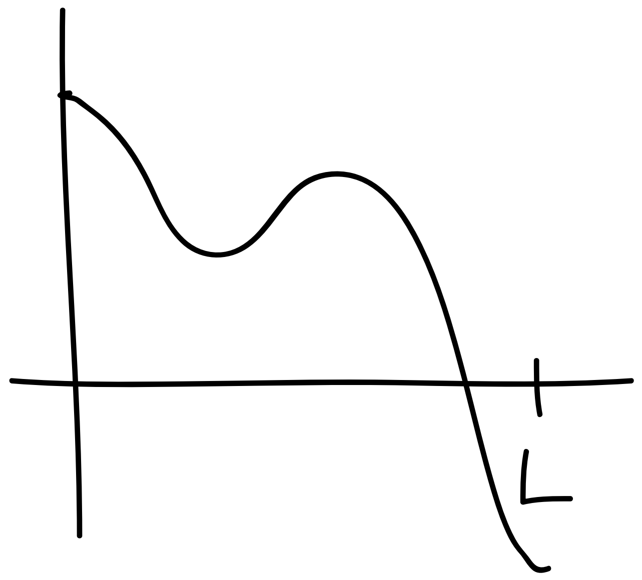
- **Theorem** Suppose f anf f' are piecewise continuous on [-L,L] and periodic beyond that interval. Then f(x) = f_{FS}(x) at all points at which f is continuous. Furthermore, at points of discontinuity, f_{FS}(x) takes the value of the midpoint of the jump. That is,

$$f_{FS}(x) = \frac{f(x^+) + f(x^-)}{2}$$

Fourier series

- Suppose you have a function on the interval [0,L] and you would like to represent it using a Fourier series. Need to make it periodic somehow. There are a few options for how to do this.



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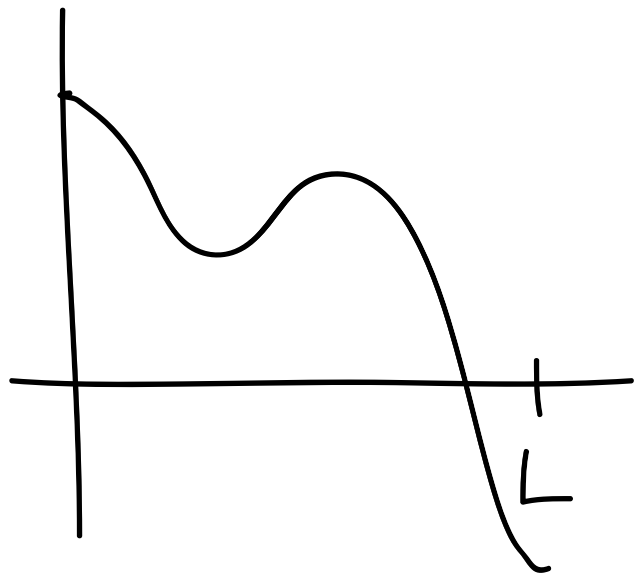
    1. Use the function given on [0,L] and extend it outside that interval so that it has period L.



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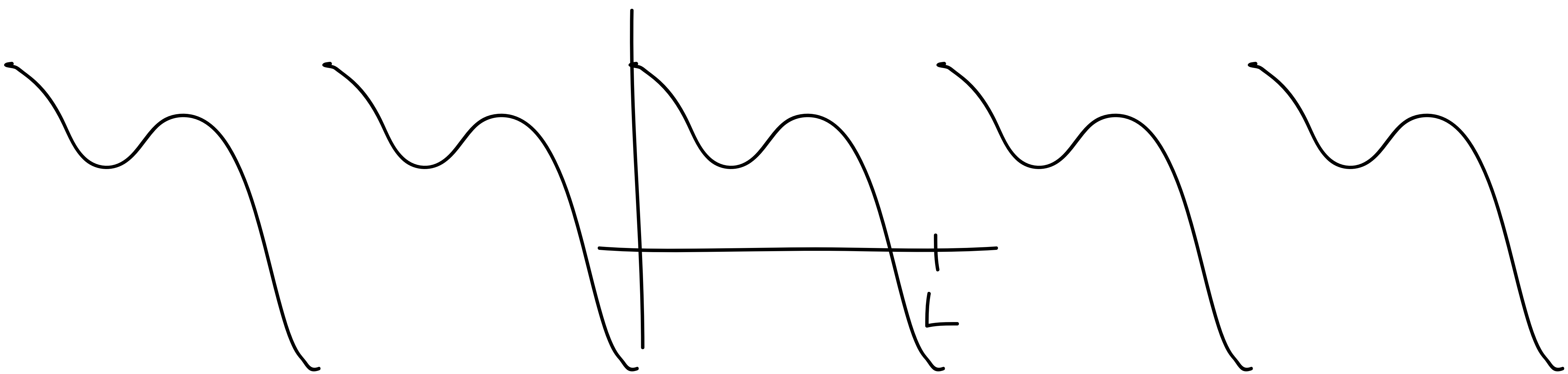
        - Is this extension even? odd?



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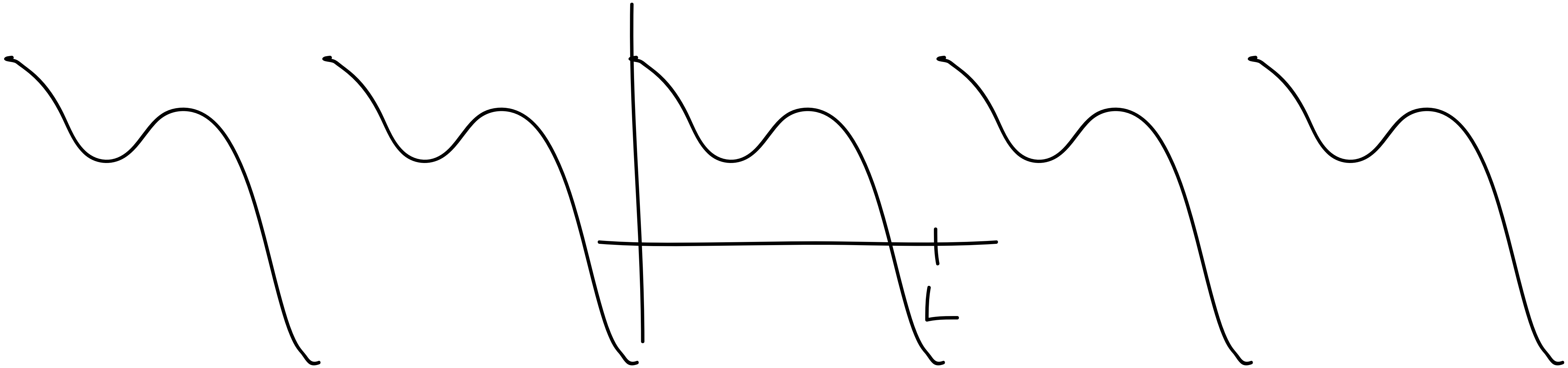

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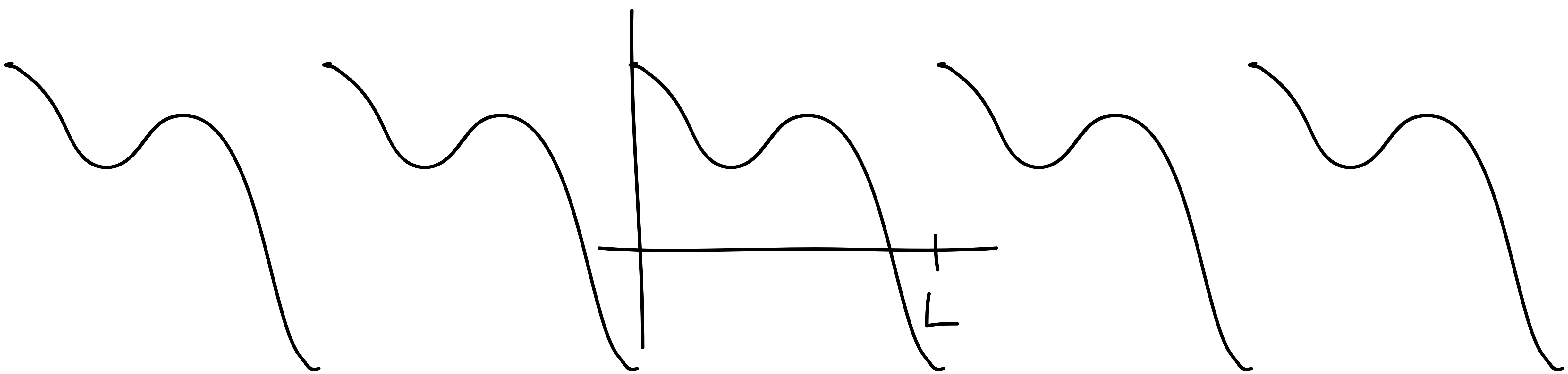

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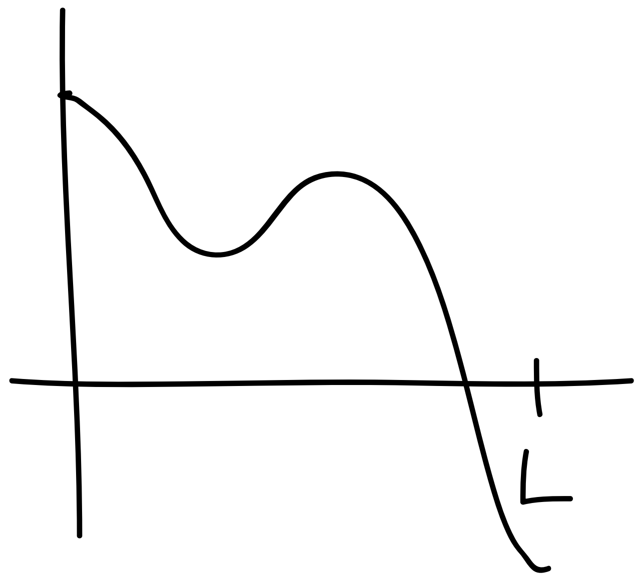
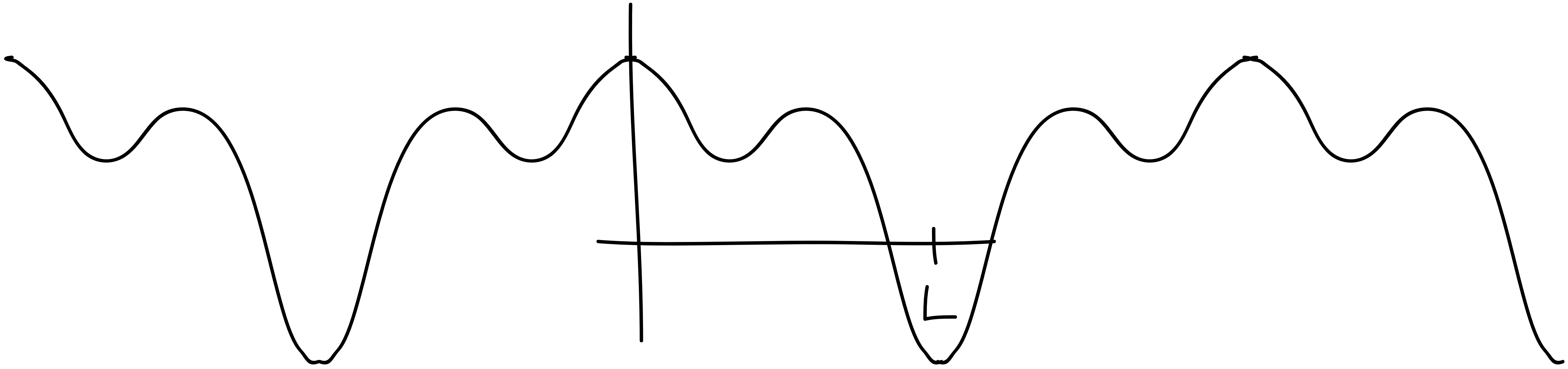
    2. Reflect it about y-axis first, then extend with period 2L.



Fourier series

- Suppose you have a function on the interval [0,L] and you would like to represent it using a Fourier series. Need to make it periodic somehow. There are a few options for how to do this.
  1. Use the function given on [0,L] and extend it outside that interval so that it has period L.
  2. Reflect it about y-axis first, then extend with period 2L.

   - Is this extension even? odd?



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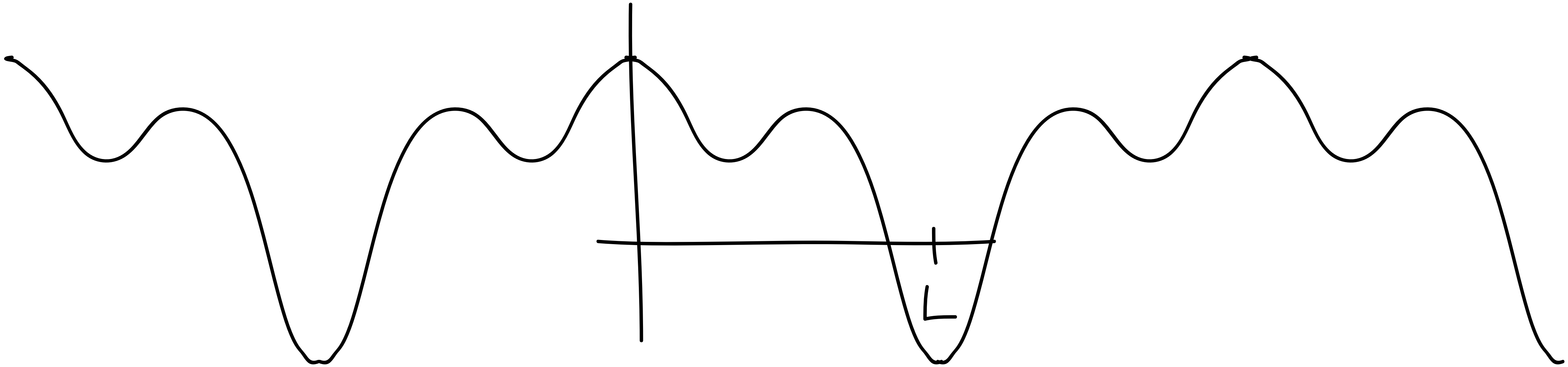

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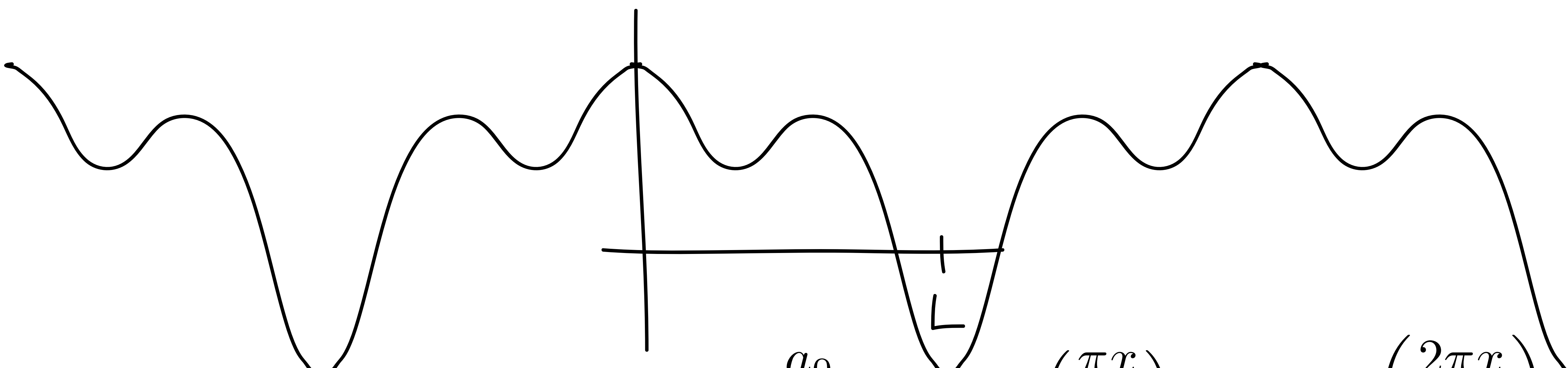

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$

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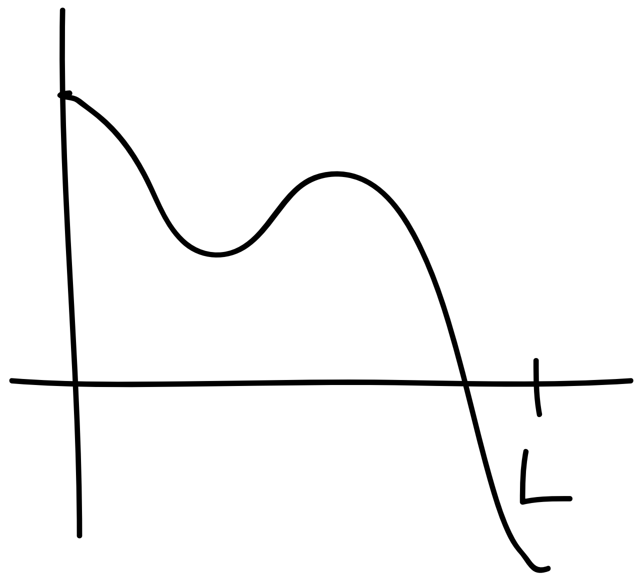

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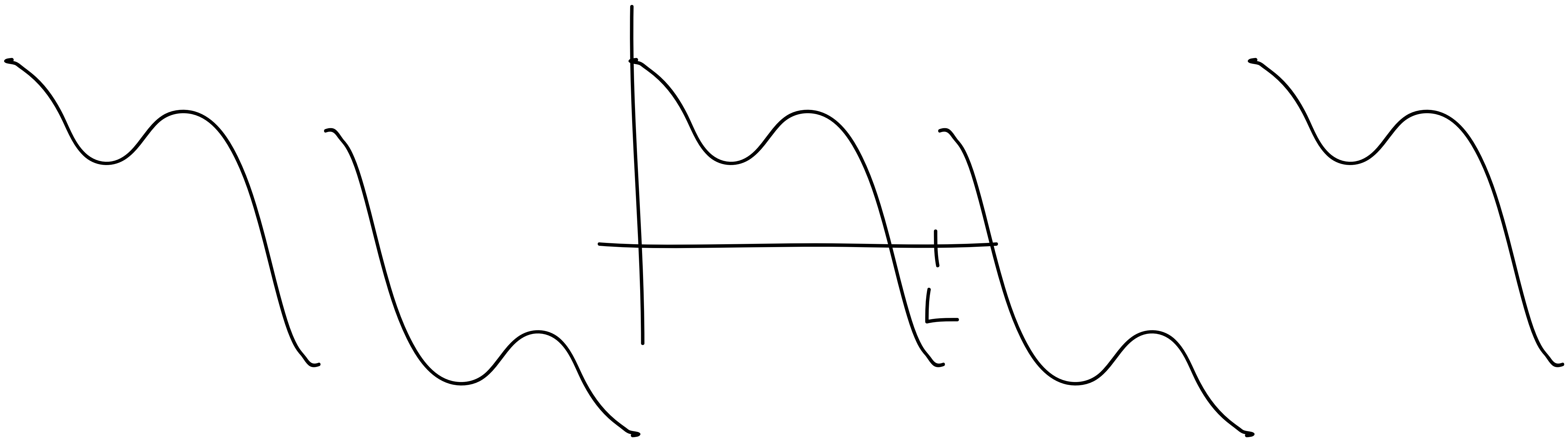

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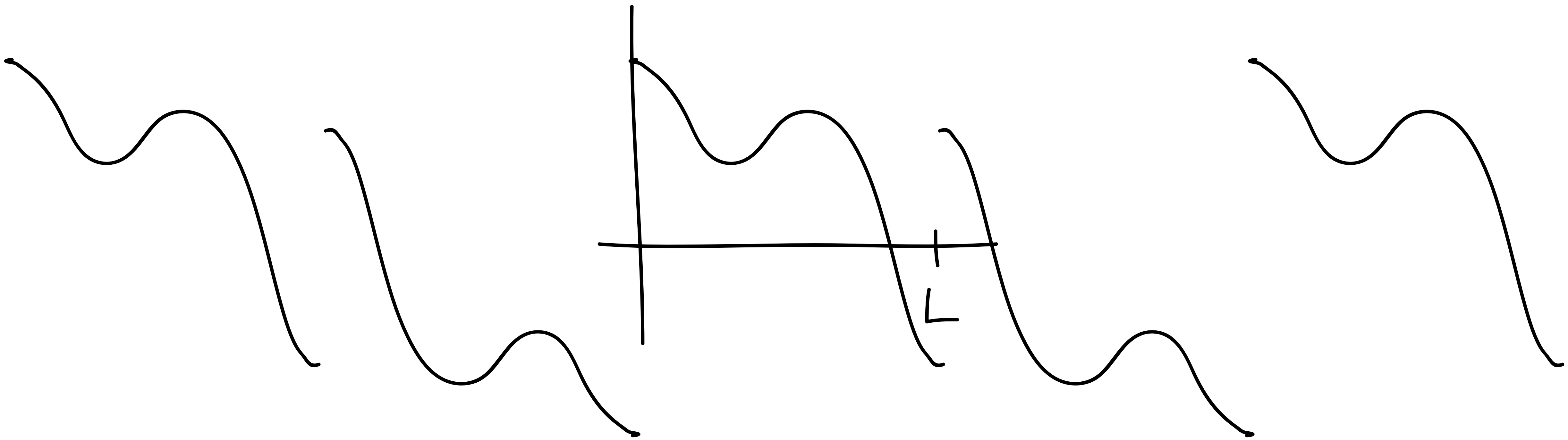

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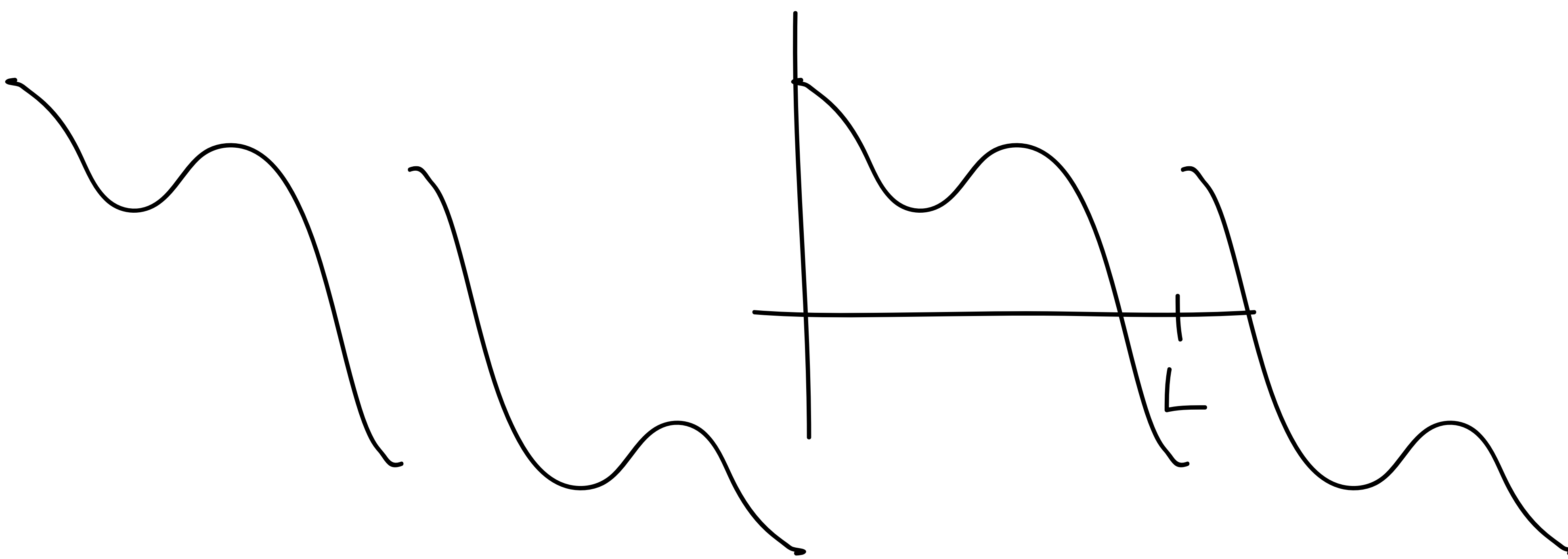

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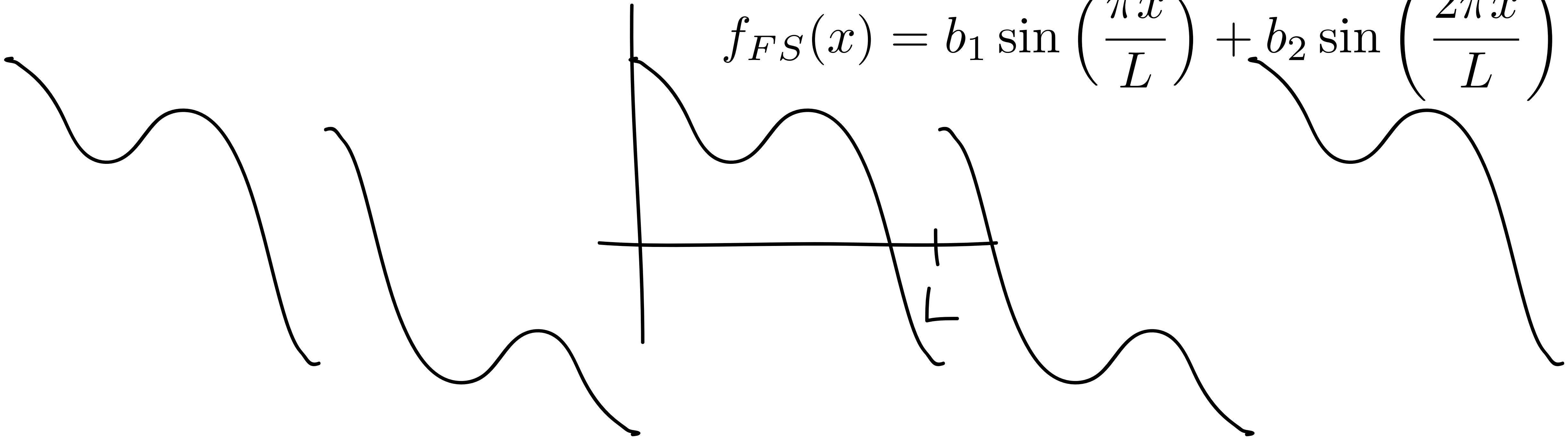
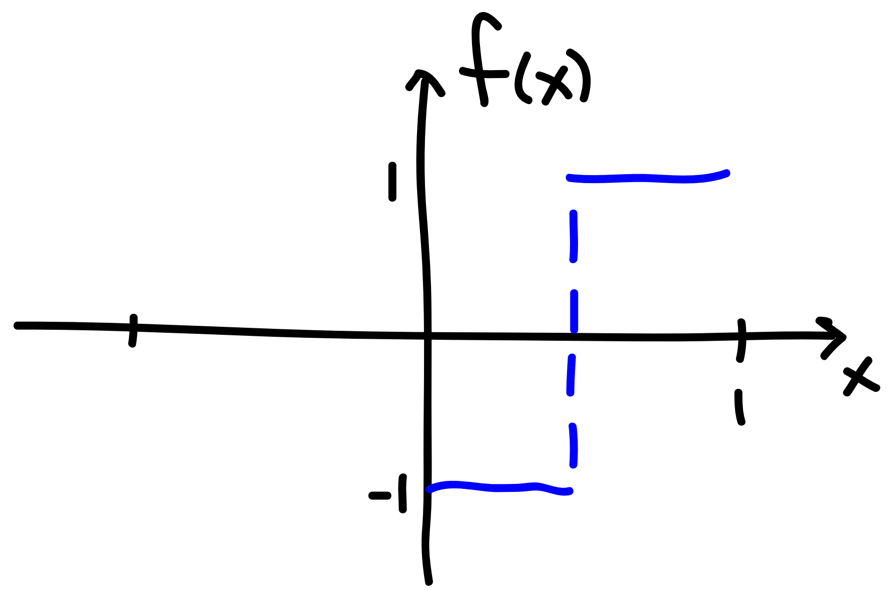
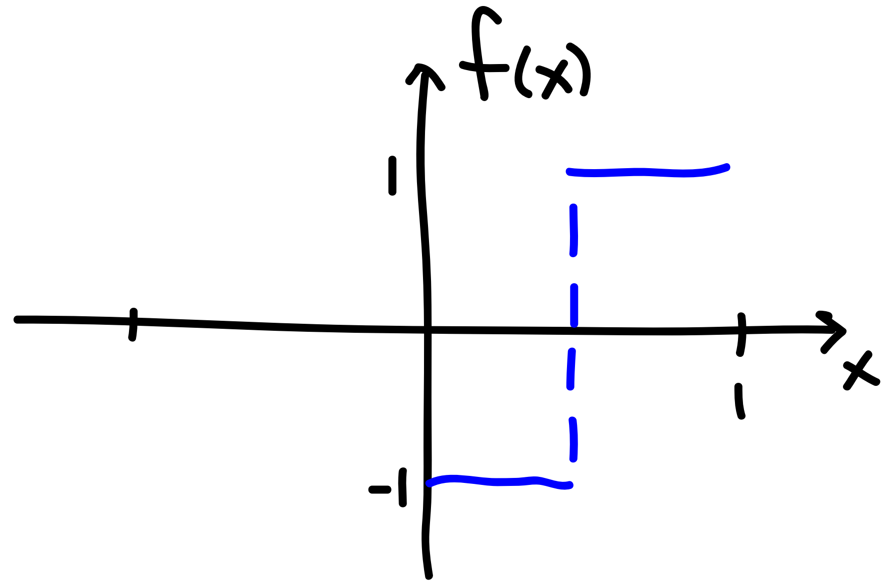
$$f_{FS}(x) = b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

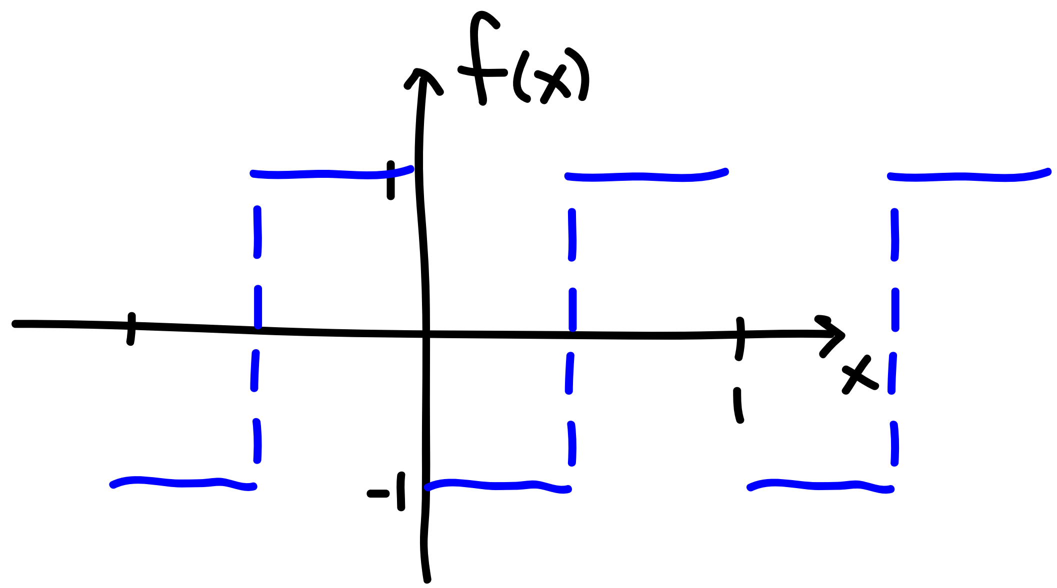


Examples



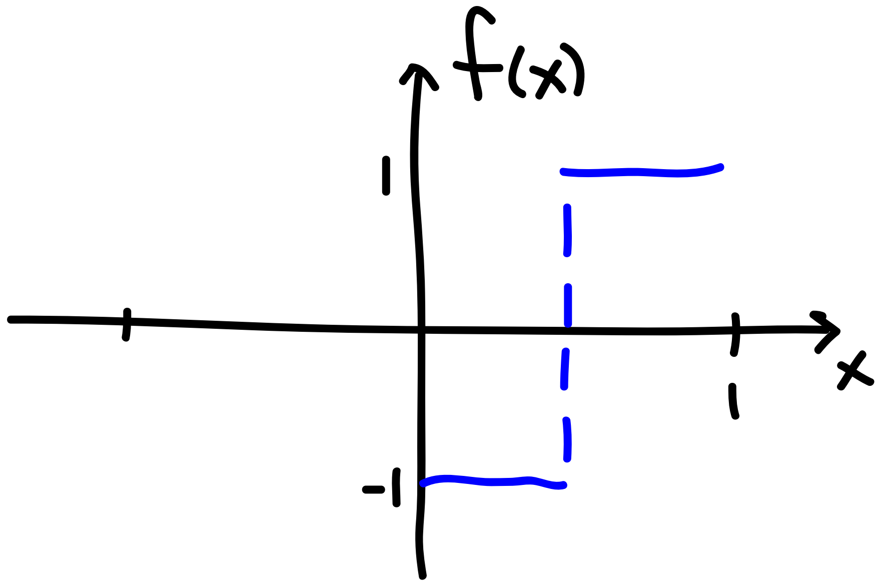
Examples

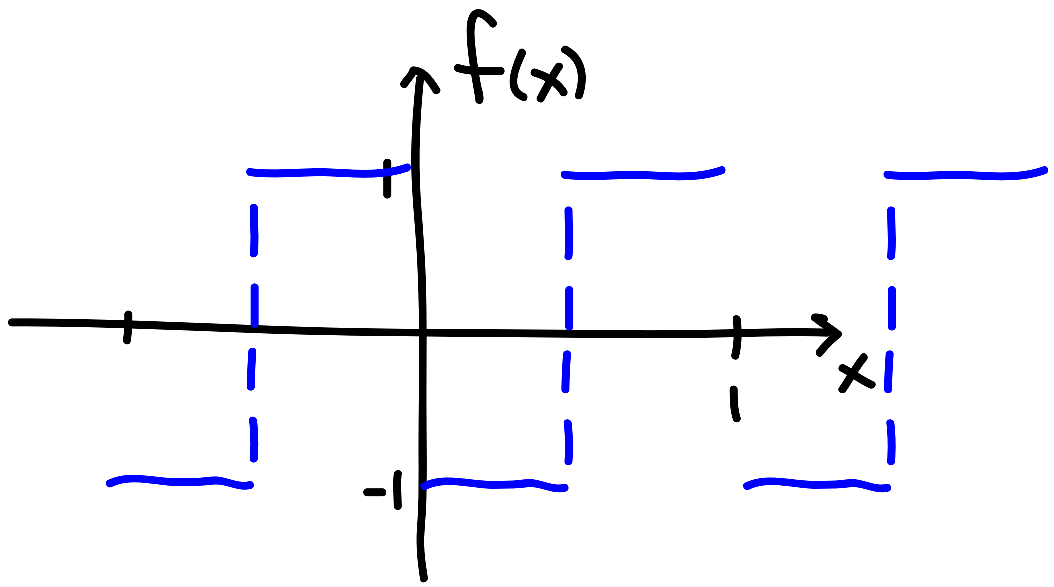




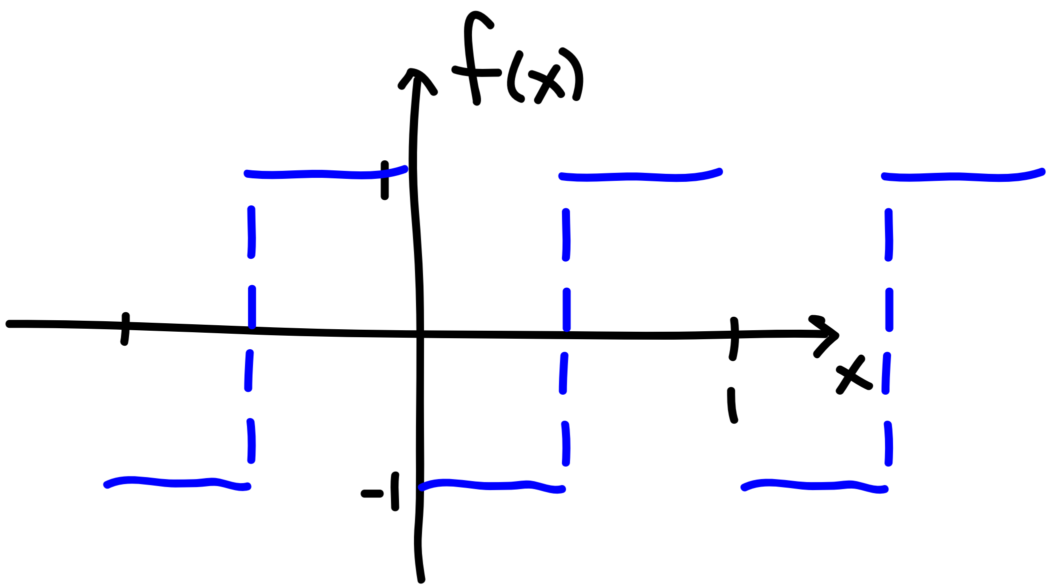
Periodic extension

Examples



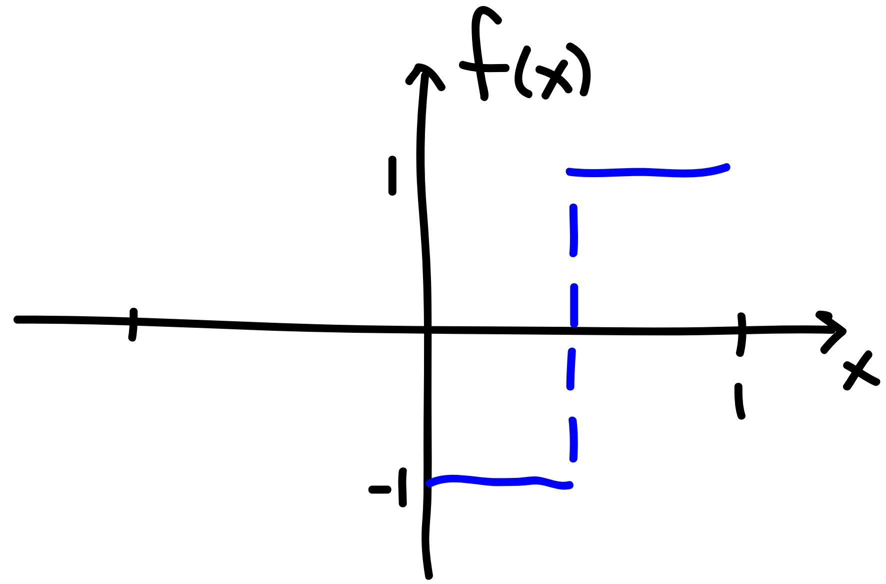


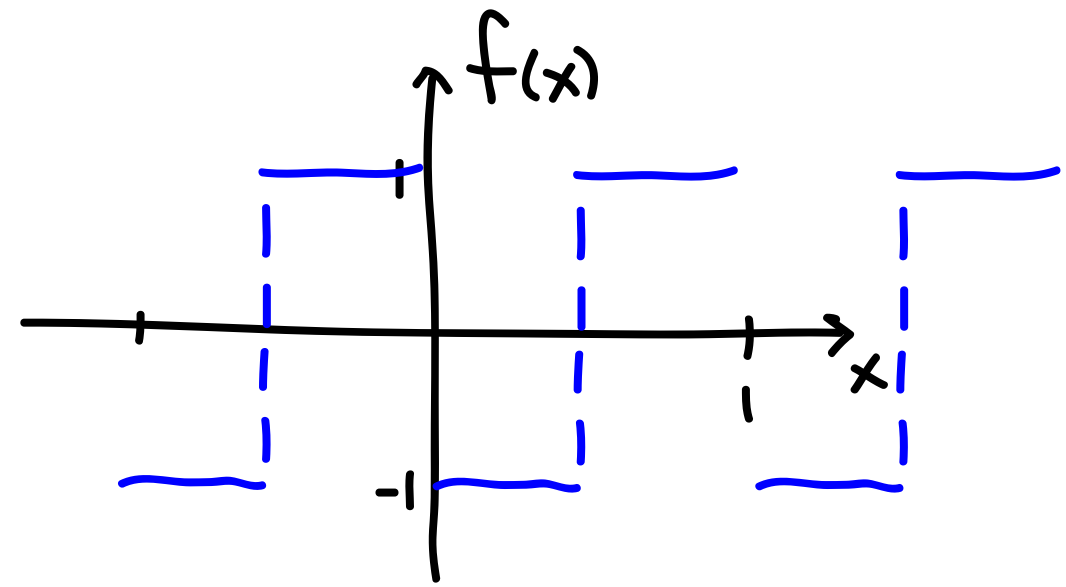
Odd periodic extension



Periodic extension

Examples





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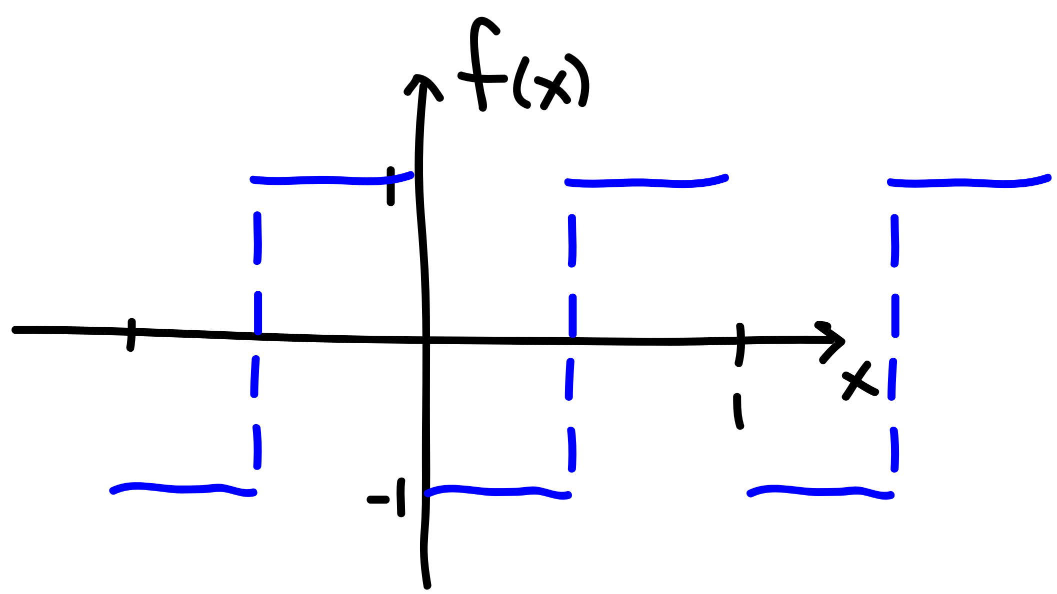
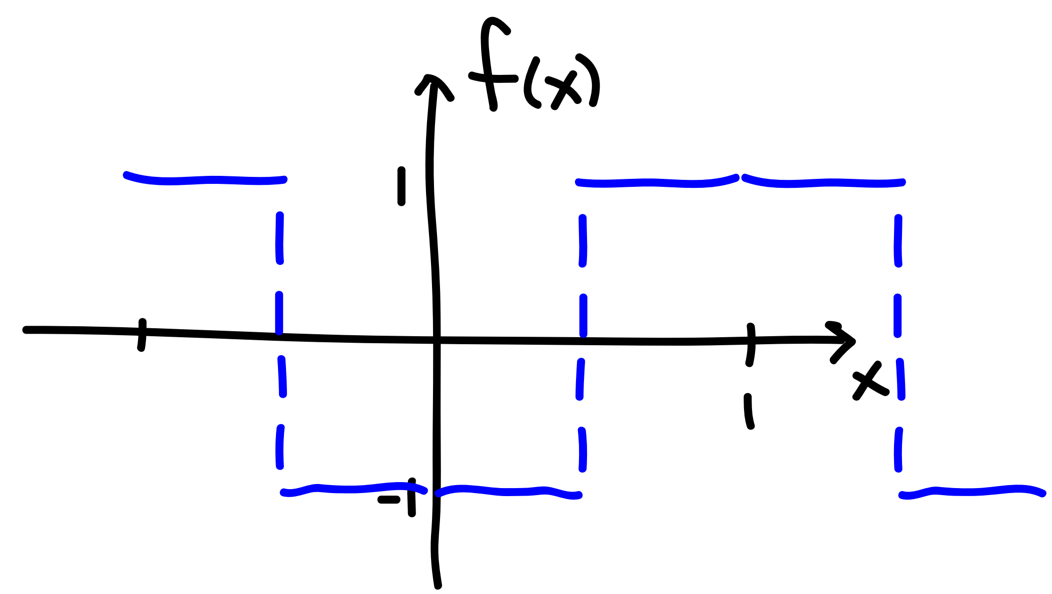


Periodic extension



Even periodic extension