Today

- General solutions, independence of functions and the Wronskian
- Distinct roots of the characteristic equation
- Review of complex numbers
- Complex roots of the characteristic equation

Modeling - Example

- Saltwater with a concentration of 200 g/L flows into a tank at a rate 2 L/min. The tank starts with no salt in it and holds 10 L. The tank is well mixed and the mixed water drains out at the same rate as the inflow.

  (a) Write down an IVP for the mass of salt in the tank as a function of time.

  (b) What is the limiting mass of salt in the tank?

(b) Directly from the equation ($m' = 400 - m/5$), find an m for which $m'=0$.

- $m=2000$. Called steady state - a constant solution.
- What happens when $m < 2000$? ---> $m' > 0$.
- What happens when $m > 2000$? ---> $m' < 0$.
- Limiting mass: 2000 g (Long way: solve the eq. and let $t \to \infty$.)

Existence and uniqueness

Theorem 2.4.2 Let the functions f and $\dfrac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta, \quad \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t_0 < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the IVP

$$y' = f(t, y), \quad y(t_0) = y_0.$$

- A couple questions/examples to explore on your own:
  - Why don't we get a solution all the way to the ends of the t interval?
    - Example: $$\frac{dy}{dt} = y^2, \quad y(0) = 1$$
  - How does a non-continuous RHS lead to more than one solution?
    - Example: $$\frac{dy}{dt} = \sqrt{y}, \quad y(0) = 0$$

Second order linear equations

- The general form for a second order linear equation:

$$y'' + p(t)y' + q(t)y = g(t)$$

- Now, an IVP requires two ICs:

$$y(0) = y_0, \quad y'(0) = v_0$$

- As with first order linear equations, we have homogeneous (g=0) and non-homogeneous second order linear equations.

- We'll start by considering the homogeneous case with constant coefficients:

$$ay'' + by' + cy = 0$$

Homogeneous equations with constant coefficients

$$ay'' + by' + cy = 0$$

- Suppose you already found a couple solutions, $y_1(t)$ and $y_2(t)$. This means that

$$ay_1'' + by_1' + cy_1 = 0 \qquad \text{and} \qquad ay_2'' + by_2' + cy_2 = 0$$

- Notice that $y(t) = C_1y_1(t)$ is also a solution. Plug it in and check:

$$\begin{aligned} &a(C_1y_1)'' + b(C_1y_1)' + c(C_1y_1) \\ &= aC_1(y_1)'' + bC_1(y_1)' + cC_1(y_1) \\ &= C_1(ay_1'' + by_1' + cy_1) = 0 \end{aligned}$$

Homogeneous equations with constant coefficients

- Which of the following functions are also solutions?

  (A) $y(t) = y_1(t)^2$

  (B) $y(t) = y_1(t)+y_2(t)$

  (C) $y(t) = y_1(t)\ y_2(t)$

  (D) $y(t) = y_1(t) / y_2(t)$

- In fact, the following are all solutions: $C_1y_1(t)$, $C_2y_2(t)$, $C_1y_1(t)+C_2y_2(t)$.
- With first order equations, the arbitrary constant appeared through an integration step in our methods. With second order equations, not so lucky.
- Instead, find two independent solutions, $y_1(t)$, $y_2(t)$, by whatever method.
- The general solution will be $y(t) = C_1y_1(t) + C_2y_2(t)$.

Homogeneous equations with constant coefficients

- One case where the arbitrary constants DO appear as we calculate:

$$y'' + y' = 0$$

$$y' + y = C_1$$

$$e^t y' + e^t y = C_1 e^t$$

$$(e^t y)' = C_1 e^t$$

$$e^t y = C_1 e^t + C_2$$

$$y = C_1 + C_2 e^{-t}$$

- More common would be that we find solutions y(t) = 1 and y(t)= e^{-t} and simply write down

$$y = C_1 + C_2 e^{-t}$$

Homogeneous equations with constant coefficients

- So in general how do we find the two independent solutions y_1 and y_2?
- Exponential solutions seem to be common so let's assume $y(t)=e^{rt}$ and see if that gets us anything useful..
- Solve $y'' + y' = 0$ by assuming $y(t) = e^{rt}$ for some constant r.

$$(e^{rt})'' + (e^{rt})' = 0$$
$$r^2 e^{rt} + r e^{rt} = 0$$
$$r^2 + r = 0$$
$$r(r+1) = 0$$
$$r = 0, \quad r = -1$$

$$y = C_1 e^0 + C_2 e^{-t}$$
$$y = C_1 + C_2 e^{-t}$$

Homogeneous equations with constant coefficients

- Solve $y'' - 4y = 0$ subject to the ICs $y(0) = 3, y'(0) = 2$.

(A) $y(t) = C_1 e^{2t} + C_2 e^{-2t}$

(B) $y(t) = 2e^{2t} + e^{-2t}$

(C) $y(t) = \frac{7}{4} e^{4t} + \frac{5}{4} e^{-4t}$

(D) $y(t) = e^{2t} + 2e^{-2t}$

(E) $y(t) = C_1 e^{4t} + C_2 e^{-4t}$

Homogeneous equations with constant coefficients

- For the general case, $ay'' + by' + cy = 0$, by assuming $y(t) = e^{rt}$ we get the characteristic equation:

$$ar^2 + br + c = 0$$

- There are three cases.

  i. Two distinct real roots: $b^2 - 4ac > 0$. ($r_1 \neq r_2$)

  ii. A repeated real root: $b^2 - 4ac = 0$.

  iii. Two complex roots: $b^2 - 4ac < 0$.

- For case i, we get $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.

- Do our two solutions cover all possible ICs? That is, can we use them to form a general solution?

Independence and the Wronskian (Section 3.2)

- Example: Suppose $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are two solutions to some equation. Can we solve ANY initial condition $y(0) = y_0,\ y'(0) = v_0$ with these two solutions?

$$y(t) = C_1 e^{2t+3} + C_2 e^{2t-3}$$

$$y(0) = C_1 e^3 + C_2 e^{-3} = y_0$$

$$y'(0) = 2C_1 e^3 + 2C_2 e^{-3} = v_0$$

- Solve this system for C_1, C_2...

- Can't do it. Why? $\begin{pmatrix} e^3 & e^{-3} \\ 2e^3 & 2e^{-3} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$

$$\det \begin{pmatrix} e^3 & e^{-3} \\ 2e^3 & 2e^{-3} \end{pmatrix} = 0$$

Independence and the Wronskian (Section 3.2)

- For any two solutions to some linear ODE, to ensure that we have a general solution, we need to check that

$$\det \begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = y_1(0)y_2'(0) - y_1'(0)y_2(0) \neq 0$$

- For ICs other than $t_0=0$, we require that

$$y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$$

- This quantity is called the Wronskian.

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

Independence and the Wronskian (Section 3.2)

- Two functions $y_1(t)$ and $y_2(t)$ are linearly independent provided that the only way that $C_1y_1(t) + C_2y_2(t) = 0$ for all values of t is when $C_1=C_2=0$.

  e.g. $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are not independent.

  Find values of $C_1 \neq 0$ and $C_2 \neq 0$ so that $C_1y_1(t) + C_2y_2(t) = 0$.

(A) $C_1 = e^{-2t-3}, \; C_2 = -e^{-2t+3}$

(B) $C_1 = e^{-2t+3}, \; C_2 = -e^{-2t-3}$

(C) $C_1 = e^{-3}, \; C_2 = e^{3}$

(D) $C_1 = e^{-3}, \; C_2 = -e^{3}$

(E) $C_1 = e^{3}, \; C_2 = -e^{-3}$

Independence and the Wronskian (Section 3.2)

- Two functions $y_1(t)$ and $y_2(t)$ are linearly independent provided that the only way that $C_1y_1(t) + C_2y_2(t) = 0$ for all values of t is when $C_1=C_2=0$.

  e.g. $y_1(t) = e^{2t+3}$ and $y_2(t) = e^{2t-3}$ are not independent.

- The Wronskian is defined for any two functions, even if they aren't solutions to an ODE.

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

- If the Wronskian is nonzero for some t, the functions are linearly independent.

- If $y_1(t)$ and $y_2(t)$ are solutions to an ODE and the Wronskian is nonzero then they are independent and

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

is the general solution. We call $y_1(t)$ and $y_2(t)$ a fundamental set of solutions and we can use them to solve any IC.

Independence and the Wronskian (Section 3.2)

- So for case i (distinct roots), can we form a general solution from

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = e^{r_2 t} \text{ ?}$$

- Must check the Wronskian:

$$W(e^{r_1 t}, e^{r_2 t})(t) = e^{r_1 t} r_2 e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t}$$

$$= (r_1 - r_2) e^{r_1 t} e^{r_2 t} \neq 0$$

So yes! $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is the general solution.

Independence and the Wronskian (Section 3.2)

- Example: Consider the equation $y'' + 9y = 0$. Find the roots of the characteristic equation (i.e. the r values).

(A) $r_1 = 3$, $r_2 = -3$.

(B) $r_1 = 3$ (repeated root).

(C) $r_1 = 3i$, $r_2 = -3i$.

(D) $r_1 = 9$, (repeated root).

As we'll see soon, this means that $y_1(t) = \cos(3t)$ and $y_2(t)=\sin(3t)$.

Do these form a fundamental set of solutions? Calculate the Wronskian.

$$W(\cos(3t), \sin(3t))(t) =$$

(A) 0

(C) 3

(B) 1

(D) $2\cos(3t)\sin(3t)$

Distinct roots - asymptotic behaviour (Section 3.1)

- Three cases:

(i) Both r values positive.

e.g. $y(t) = C_1 e^{2t} + C_2 e^{5t}$

(ii) Both r values negative.

e.g. $y(t) = C_1 e^{-2t} + C_2 e^{-5t}$

(iii) The r values have opposite sign.

e.g. $y(t) = C_1 e^{-2t} + C_2 e^{5t}$

Except for the zero solution y(t)=0, the limit $\lim_{t \to \infty} y(t)$ ...

(A) ...is unbounded for all ICs.

(B) ...is unbounded for most ICs but not for a few carefully chosen ones.

(C) ...goes to zero for all ICs.

Challenge: come up with an initial condition for (iii) that has a bounded solution.

Complex roots (Section 3.3)

- Complex number review (Euler's formula)
- Complex roots of the characteristic equation
- From complex solutions to real solutions

Complex number review

- We define a new number: $i = \sqrt{-1}$

- Before, we would get stuck solving any equation that required square-rooting a negative number. No longer.

- e.g. The solutions to $x^2 - 4x + 5 = 0$ are $x = 2 + i$ and $x = 2 - i$

- For any equation, $ax^2 + bx + c = 0$, when $b^2 - 4ac < 0$, the solutions have the form $x = \alpha \pm \beta i$ where α and β are both real numbers.

- For α+βi, we call α the real part and β the imaginary part.

Complex number review

- Adding two complex numbers:

$$(a + bi) + (c + di) = \underbrace{a + c}_{} + \underbrace{(b + d)}_{}i$$

- Multiplying two complex numbers:

$$(a + bi)(c + di) = \underbrace{ac - bd}_{} + \underbrace{(ad + bc)}_{}i$$

- Dividing by a complex number:

$$(a + bi)/(c + di) = (a + bi)\frac{1}{(c + di)}$$

  - What is the inverse of c+di?

Complex number review

- What is the inverse of c+di written in the usual complex form p+qi?

(A) $c - di$

(B) $\dfrac{c+di}{c^2+d^2}$

★(C) $\dfrac{c-di}{c^2+d^2}$

(D) $\dfrac{1}{c-di}$

$$(c+di)\frac{c-di}{c^2+d^2} = \frac{c^2+d^2-(cd-dc)i}{c^2+d^2} = 1$$

- Dividing by a complex number:

$$(a+bi)/(c+di) = (a+bi)\frac{c-di}{c^2+d^2} = \underline{\frac{ac+bd}{c^2+d^2}} + \underline{\frac{(bc-ad)i}{c^2+d^2}}$$

Complex number review

- Definitions:
  - Conjugate - the conjugate of $a + bi$ is

$$\overline{a + bi} = a - bi$$

  - Magnitude - the magnitude of $a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}$$