Today

- Method of Undetermined Coefficients for any periodic function.
- Fourier Series and method of undetermined coefficients
- Fourier series calculations

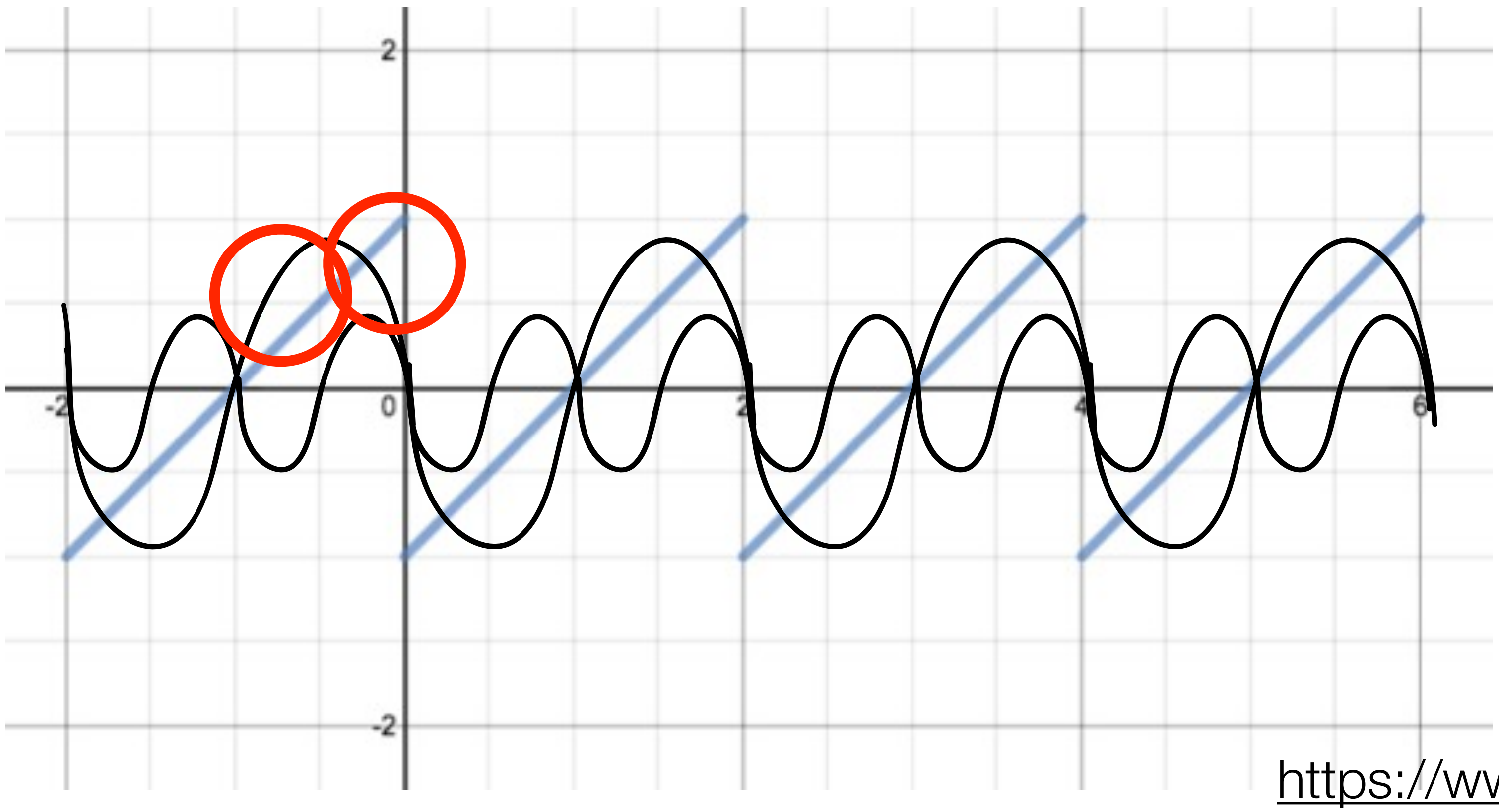
Fourier series

- Recall Method of Undetermined Coefficients for equations of the form

$$ay'' + by' + cy = f(t)$$

- Applicable for functions f(t) that are polynomials, exponentials, sin, cos and products of those.

- How about functions like this (periodic but not trig)?



- What if we could construct such functions using only sine and cosine functions?

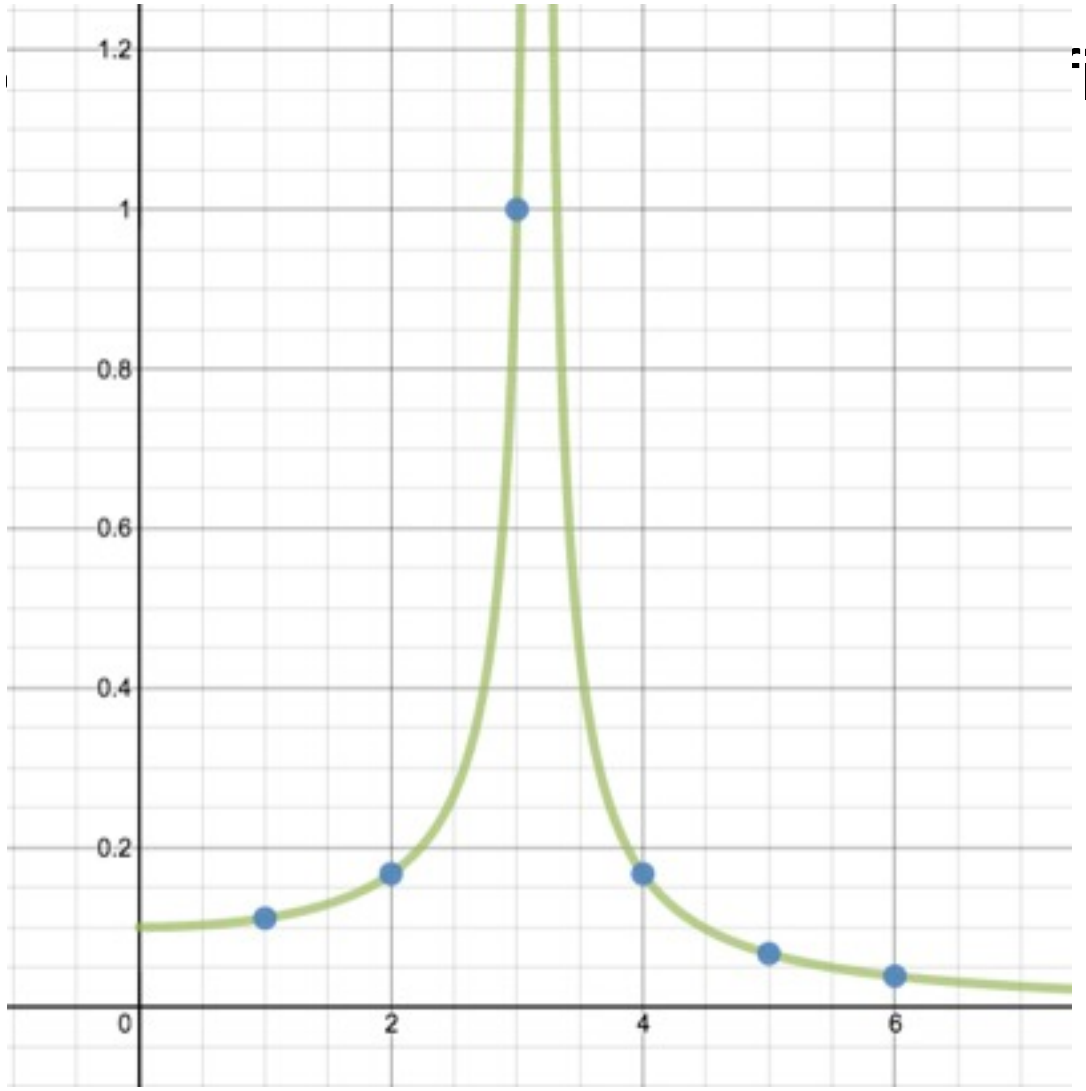
https://www.desmos.com/calculator/tqaia8bt7n

Fourier series

- For the equation

$$y'' + 10y = \cos(t) + \frac{1}{2}\cos(2t) + \frac{1}{3}\cos(3t) + \frac{1}{4}\cos(4t) + \cdots$$

- what will be the ficient) in the solution?



(A) w = 1

(B) w = 2

(C) w = 3

(D) w = 4

(E) Don’t know. Explain please.

$$A(\omega) = \frac{1}{|\omega_0^2 - \omega^2|}$$

Fourier series

- Replace f(t) by a sum of trig functions, if possible:

$$ay'' + by' + cy = f(t) \stackrel{?}{=} A_0 + \sum_{n=1}^{N} a_n \cos(\omega_n t) + \sum_{n=1}^{N} b_n \sin(\omega_n t)$$

- For any f(t), how do we find the best choice of A_0, a_n, b_n?

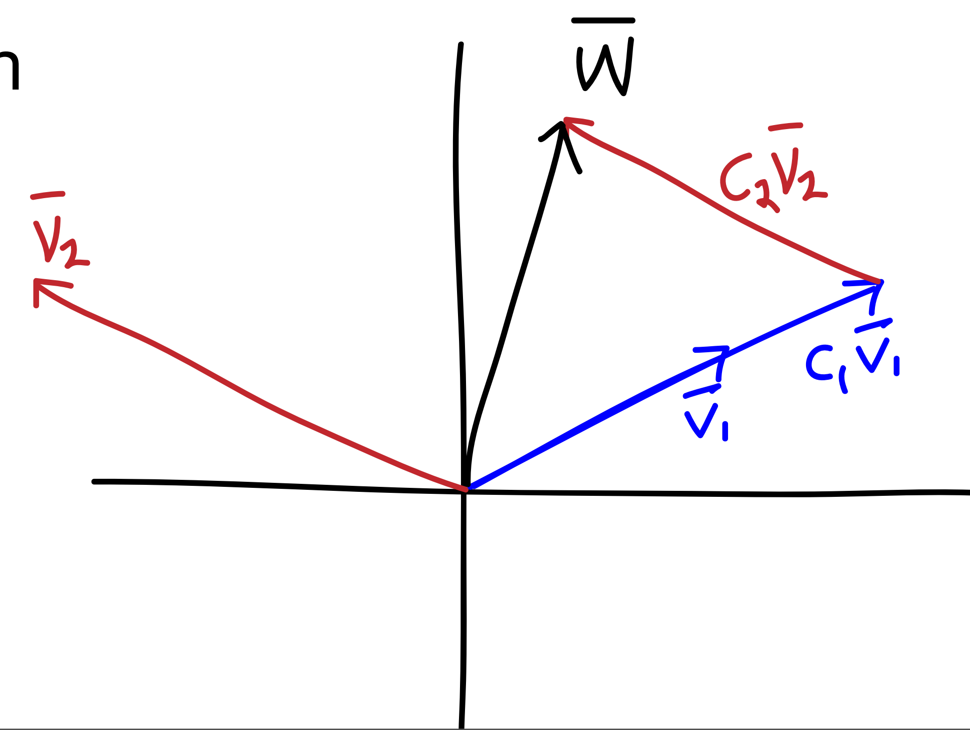
- This problem is closely related to an analogous vector problem: how do you choose c_1, c_2 so that $w = c_1 v_1 + c_2 v_2$?

- If v_1 and v_2 are perpendicular ($v_1 \circ v_2 = 0$), then

$$\mathbf{w} \circ \mathbf{v_1} = c_1 \mathbf{v_1} \circ \mathbf{v_1} + c_2 \mathbf{v_2} \circ \mathbf{v_1}$$

$$c_1 = \frac{\mathbf{w} \circ \mathbf{v_1}}{\mathbf{v_1} \circ \mathbf{v_1}}$$

$$\mathbf{v_1} \circ \mathbf{v_1} = ||\mathbf{v_1}||^2 \qquad c_2 = \frac{\mathbf{w} \circ \mathbf{v_2}}{\mathbf{v_2} \circ \mathbf{v_2}}$$



Fourier series

- For functions, define dot product as

$$g(x) \circ h(x) = \int_{\text{one period}} g(x)h(x)\ dx$$

- just like for vectors but indexed over all x instead of 1, 2, 3:

$$\mathbf{v} \circ \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Fourier series

- Back to our ODE, what do we choose for the w_n if f(t) has period T? Keep in mind that we want all the functions involved to have period T.

$$ay'' + by' + cy = f(t) \stackrel{?}{=} A_0 + \sum_{n=1}^{N} a_n \cos(\omega_n t) + \sum_{n=1}^{N} b_n \sin(\omega_n t)$$

(A) $w_n = \pi / T$

(B) $w_n = 2\pi / T$

(C) $w_n = n\pi / T$

(D) $w_n = 2\pi n / T$

(E) Don't know. Explain please.

Once we find the coefficients, this will be the Fourier series representation of f(t).

For FS in general, people use $w_n = \pi n / T$ for reasons that will make more sense once we cover PDEs.

Draw graphs on doc cam.

Finding the Fourier series coefficients

- Define $v_0(x) = 1$ $v_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ $n = (0,)1, 2, 3, \ldots$

$\ldots$

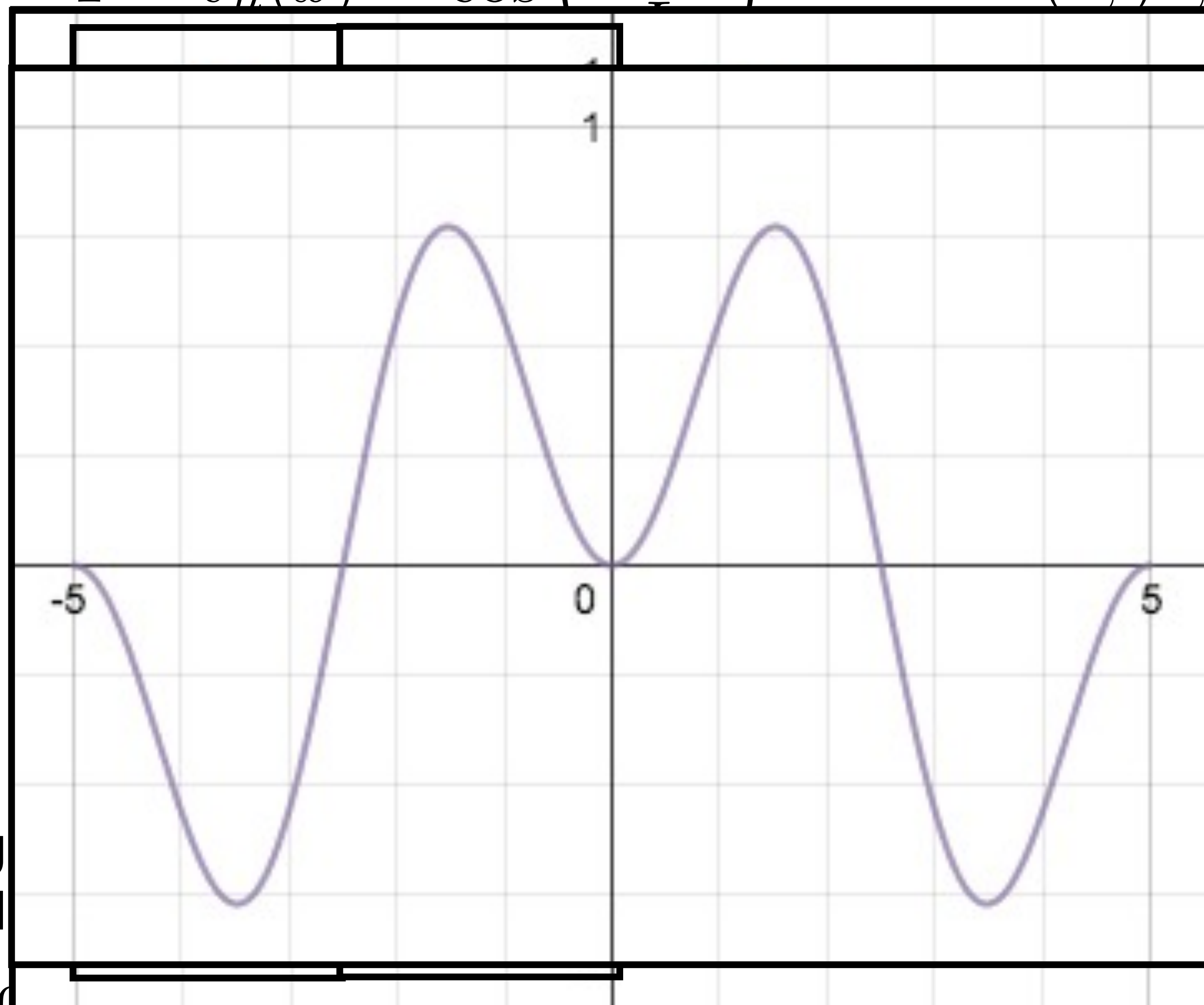
$v_0 \circ v_n =$ $(m \neq n)$

★(A) 0

(B) π

(C) π/2

(D) nπ/2



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Integral of a trig
over one period

$v_0 \circ w_n = 0$

$$v_n \circ v_n = \int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right)\, dx = L$$

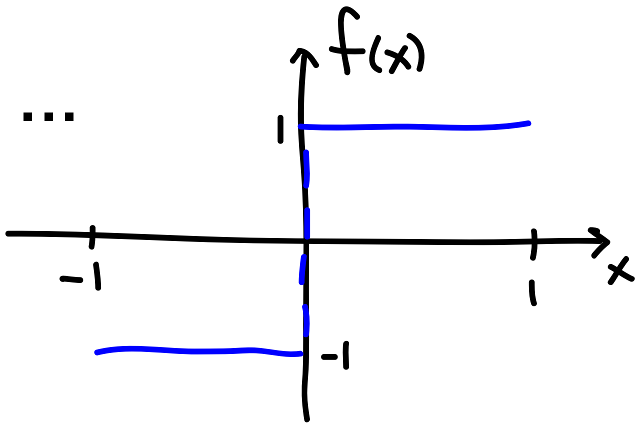
$$g(x) \circ h(x) = \int_{-L}^{L} g(x)h(x)\, dx$$

Fourier series

- Defining Fourier series:
- Define a function $f_{FS}(x)$ on the interval [-L,L] by choosing coefficients A_0, a_n and b_n and setting

$$f_{FS}(x) = A_0 + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$
$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

- This is called a Fourier series. It may or may not converge for different values of x, depending on the choice of coefficients.
- Given any function f(x) on [-L,L], can it be represented by some $f_{FS}(x)$?
- Let's check for $f(x) = 2u_0(x)-1$ on the interval [-1,1] ...

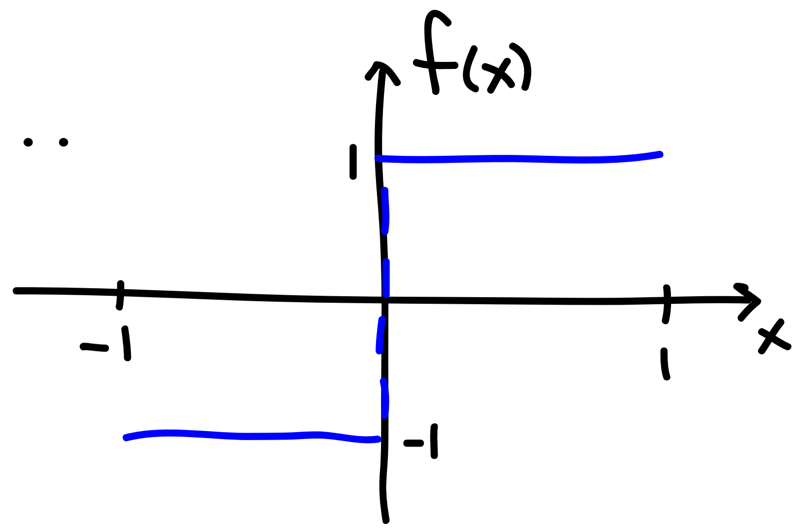


Fourier series

- Find the Fourier series for f(x) = $2u_0(x)-1$ on the interval [-1,1].

$$f_{FS}(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$

$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$



- Our hope is that f(x) = $f_{FS}(x)$ so we calculate coefficients as if they were equal:

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

A_0 is the average value of f(x)!

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\ dx$$

- To simplify formulas, usually define

$$a_0 = 2A_0 = \frac{1}{L}\int_{-L}^{L} f(x)\ dx$$