Today

- Forced vibrations
  - Newton's 2nd Law with external forcing.
  - Forced mass-spring system without damping away from resonance.
  - Forced mass-spring system without damping at resonance.
  - Forced mass-spring system with damping.
- Midterm (Feb 10, in class) - Everything up to and including Monday Feb 3 (systems of equations and review of eigenvectors).

Forced vibrations (3.8)

- Newton's 2nd Law:

$$ma = -kx - \gamma v \ + F(t)$$

Forced vibrations (3.8)

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Forced vibrations (3.8)

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Forced vibrations (3.8)

- Newton's 2nd Law:

$$ma = -kx - \gamma v + F(t)$$

spring force

drag force

applied/external force

Forced vibrations (3.8)

- Newton's 2nd Law:

$$\underbrace{ma = -kx}_{\text{spring force}} \; \underbrace{- \gamma v}_{\text{drag force}} \; \underbrace{+F(t)}_{\text{applied/external force}}$$

$$mx'' + \gamma x' + kx = F(t)$$

Forced vibrations (3.8)

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- Forced vibrations - nonhomogeneous linear equation with constant coefficients.

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- Forced vibrations - nonhomogeneous linear equation with constant coefficients.

- Building during earthquake, tuning fork near instrument, car over washboard road, polar bond under EM stimulus (classical, not quantum).

Forced vibrations (3.8)

- Without damping ($\gamma = 0$).

forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

- For what value(s) of *w* does this equation equation have an unbounded solution?

(A) w = sqrt(k/m)

(B) w = m/F_0

(C) w = $(k/m)^2$

(D) w = 2π

Forced vibrations (3.8)

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Forced vibrations (3.8)

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$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

$$x_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

Forced vibrations (3.8)

- Without damping ($\gamma = 0$).

forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

$$x_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \qquad \omega_0 = ?$$

Forced vibrations (3.8)

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forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

$$x_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

Forced vibrations (3.8)

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natural frequency

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forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

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natural frequency

- Case 1: $\omega \neq \omega_0$

Forced vibrations (3.8)

- Without damping ($\gamma = 0$).

forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

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- Case 1: $\omega \neq \omega_0$

$$x_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

natural frequency

Forced vibrations (3.8)

- Without damping ($\gamma = 0$).

forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

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- Case 1: $\omega \neq \omega_0$

natural frequency

$$x_p(t) = A \cos(\omega t) + B \sin(\omega t)$$

$$A = ?, B = ?$$

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forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

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$$x_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

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$$x_p''(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$$

$$mx_p'' + kx_p = (k - \omega^2 m)A\cos(\omega t) + (k - \omega^2 m)B\sin(\omega t)$$

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$$x_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

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$$mx_p'' + kx_p = (k - \omega^2 m)A\cos(\omega t) + (k - \omega^2 m)B\sin(\omega t)$$

$$= F_0 \cos(\omega t) \Rightarrow A = \frac{F_0}{(k - \omega^2 m)}$$

Forced vibrations (3.8)

- Without damping ($\gamma = 0$).

forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

$$x_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

natural frequency

- Case 1: $\omega \neq \omega_0$

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$$= F_0\cos(\omega t) \quad \Rightarrow A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

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Forced vibrations (3.8)

- Without damping ($\gamma = 0$), $\omega \neq \omega_0$.
  - Beats - long term behaviour includes both x_h and x_p
  - On the board.

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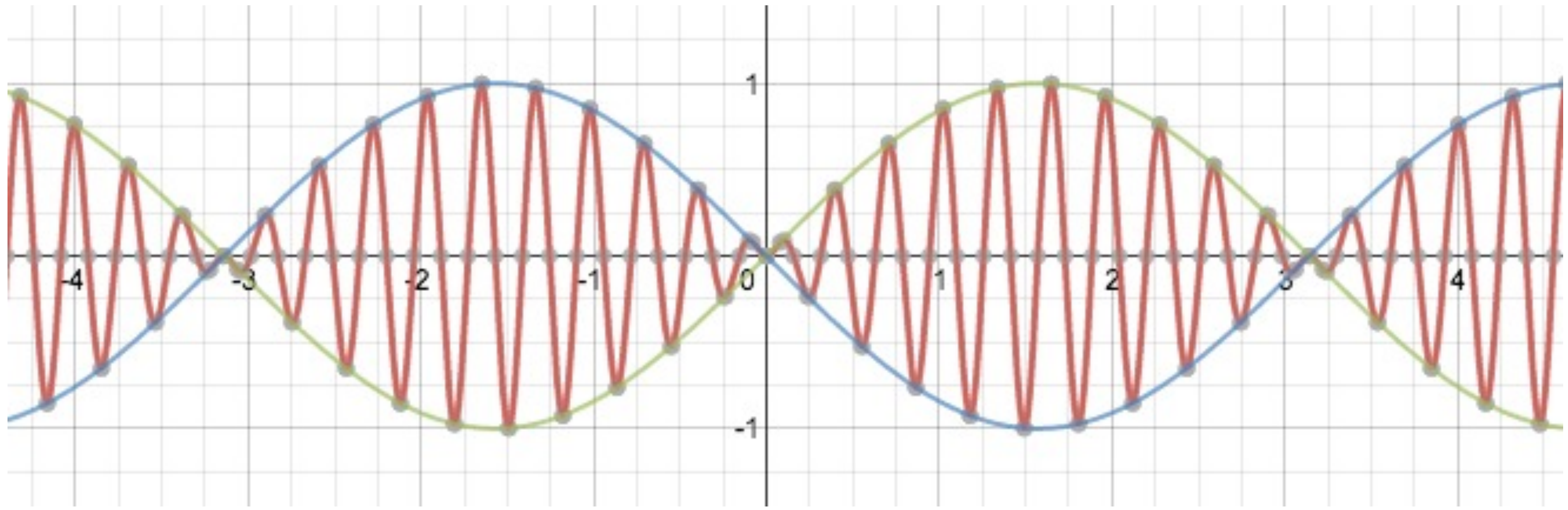

Forced vibrations (3.8)

- Case 2: $\omega = \omega_0$

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$$mx'' + kx = F_0 \cos(\omega_0 t) \qquad\qquad \omega_0 = \sqrt{\frac{k}{m}}$$

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- Case 2: $\omega = \omega_0$

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega_0 t) \qquad\qquad \omega_0 = \sqrt{\frac{k}{m}}$$

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Forced vibrations (3.8)

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$$A = 0$$

Forced vibrations (3.8)

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$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

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$$A = 0$$

$$B = \frac{F_0}{2\omega_0 m} = \frac{F_0}{2\sqrt{km}}$$

Forced vibrations (3.8)

- Case 2: $\omega = \omega_0$

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

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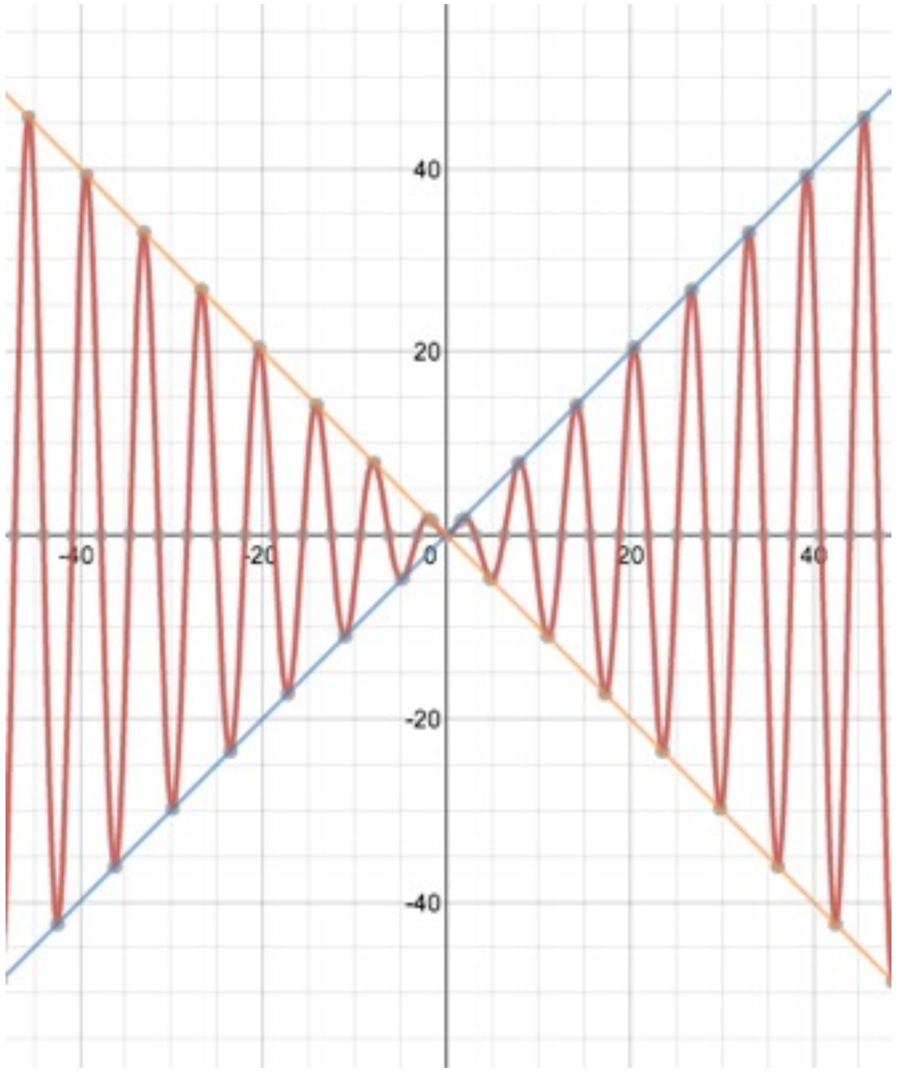
$$A = 0$$

$$B = \frac{F_0}{2\omega_0 m} = \frac{F_0}{2\sqrt{km}} \qquad x_p(t) = \frac{F_0}{2\sqrt{km}} t\sin(\omega_0 t)$$

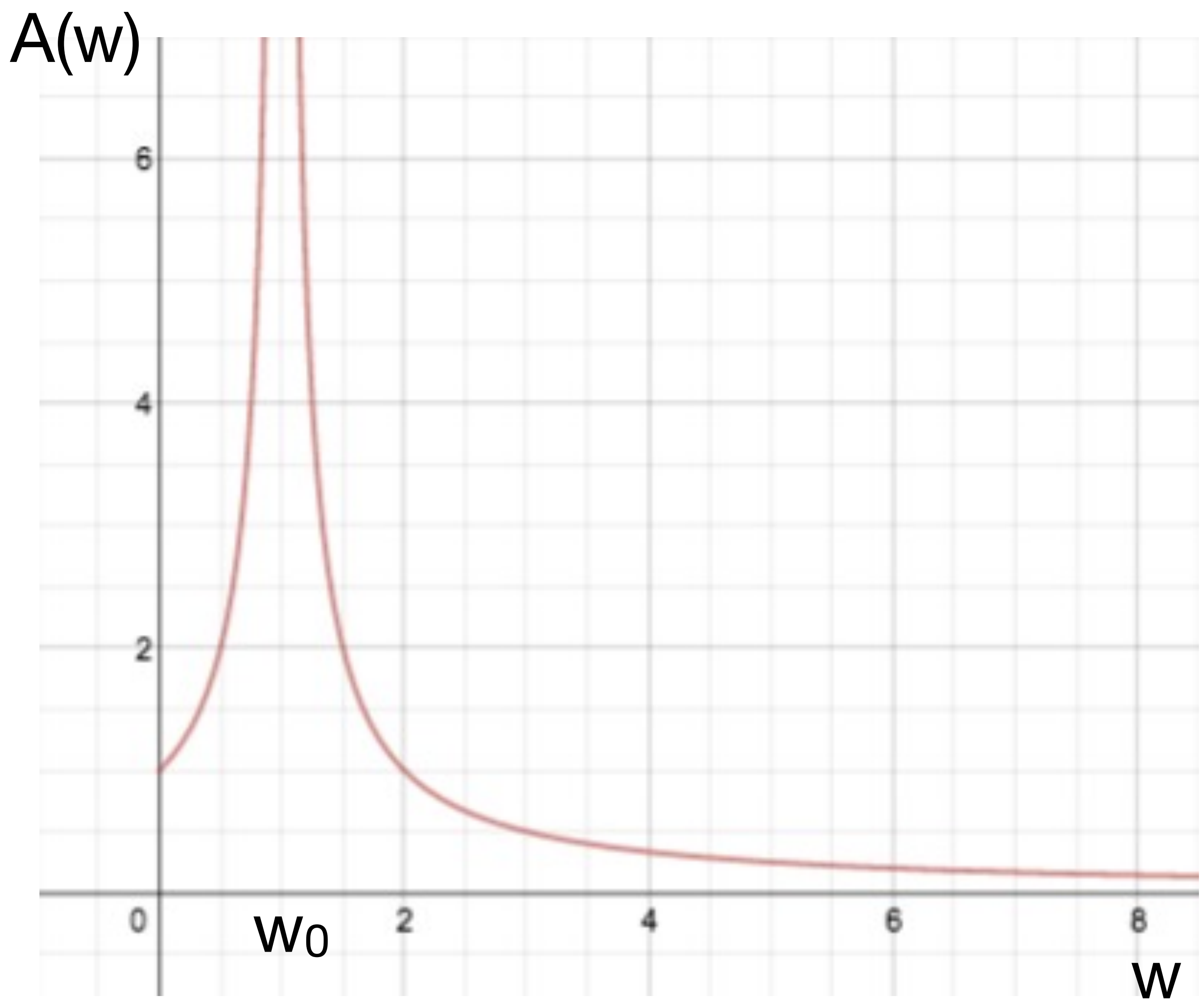
Forced vibrations (3.8)

- Without damping ($\gamma = 0$), $\omega \neq \omega_0$.
    - Long term behaviour - x_p grows unbounded, swamping out x_h.



Forced vibrations (3.8)

- Plot of the amplitude of the particular solution as a function of ω.



- Calculated:

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

- Plotted with:

$$\frac{F_0}{m} = 1, \;\; w_0 = 1$$

$$A(\omega) = \frac{1}{|\omega_0^2 - \omega^2|}$$

- Recall that for $\omega = \omega_0$, the amplitude grows without bound.

Forced vibrations (3.8)

- With damping (on the blackboard)
- Desmos illustration