MATH 256 – Midterm 1 solutions – February 10, 2015.

Last name: ____________ First name: ____________ Student #: ____________

Place a box around each answer so that it is clearly identified. Point values are approximate and may differ slightly in the final marking scheme.

1. **[5 pts]** The solution to the equation $y'' + \alpha y' + \beta y = 0$ subject to initial conditions $y(0) = \gamma, y'(0) = \delta$ is given by $y(t) = e^{-3t}(4\cos(2t) + 2\sin(2t))$. What are the constants α, β and γ? You don't need to calculate δ.

$r = -3 \pm 2i$ 2

$(r+3+2i)(r+3-2i)$ 1

$(r+3)^2 - (2i)^2$

$r^2 + 6r + 9 + 4$

$\boxed{\alpha = 6, \ \beta = 13}$ 1

$y(0) = e^0(4\cos 0 + 2\sin 0) = 4$

$\gamma = 4$ 1

2. **[5 pts]** The equation for the motion of a mass spring system is $y'' + 6y' + 9y = f(t)$. Complete each $y_p(t)$ – $f(t)$ pair in the table below. That is, for each given $f(t)$, fill in the best form for a proposed $y_p(t)$ (do not calculate the unknown coefficients) and, for each given $y_p(t)$, give a function $f(t)$ for which you would use the given $y_p(t)$ form to find the particular solution.

$y_p(t)$	$f(t)$
Ae^{3t}	Ce^{3t} (any value $C \neq A$ is ok)
$(At^2 + Bt + C)e^{3t}$	$2t^2e^{3t}$
$A\cos(\omega t) + B\sin(\omega t)$	$4\cos\omega t$
At^2e^{-3t}	Ce^{-3t} (any value $C \neq A$ is ok)
$(At^3 + Bt^2)e^{-3t}$	te^{-3t}

3. **[7 pts]** Using Reduction of Order, find a second solution to the equation $y''+y=0$ given that $y_1(t)=\sin(t)$ is a solution.

$y''+y=0 \quad , \quad y_1(t)=\sin t$

$y_2 = v(t)\sin t$ ①

$y_2' = v'(t)\sin t + v(t)\cos t$

$y_2'' = v''(t)\sin t + v'(t)\cos t + v'(t)\cos t - v(t)\sin t$

$y_2''+y_2 = v''\sin t + 2v'(t)\cos t = 0$ ①

$w = v'$

$w'\sin t + 2w\cos t = 0$

$-\dfrac{w'}{2w} = \dfrac{\cos t}{\sin t}$ ①

$-\frac{1}{2}\ln|w| = \ln|\sin t| + A$ ①

$|w|^{-1/2} = B|\sin t| \qquad (B=e^A)$

$|w|^{-1} = B^2\sin^2 t$

$v' = w = \pm\dfrac{1}{B^2\sin^2 t} = \dfrac{C}{\sin^2 t}$ ①

$v = \int \dfrac{C}{\sin^2 t}\,dt = C\cot t + D$ ①

$y_2 = (C\cot t + D)\sin t = C\cos t + D\sin t$ ①

4. **[2 pts - modified]** Suppose that

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$$

is the general solution to some system of first order differential equations. Consider the initial condition $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$. Characterize all pairs of values of a and b for which $\lim_{t\to\infty} x_1(t) = 0$.

$$\begin{pmatrix} a \\ b \end{pmatrix} = c \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad 2$$

or $a = -b$

or equivalent.

5. **[5 pts]** During a marathon, Onyin drinks gatorade at an average rate of 0.3 L/hour. Gatorade has a salt concentration of 450 mg/L. Her bladder removes fluid from her system at a rate of 0.3 L/hour and the salt concentration in that fluid is the same as the salt concentration in her system. Assume that Onyin's system consists effectively of 30 L of fluid.

   (a) Write down a differential equation for the total mass of salt $m(t)$ in Onyin's system as a function of time. You do not need to solve the equation.

$$\frac{dM}{dt} = \underset{1}{135} - \overset{1}{\frac{0.3}{30}} M \quad 1$$

$$= 135 - \frac{1}{100} M$$

   (b) Write down a differential equation for the total mass of salt $m(t)$ in Onyin's system assuming that she drinks pure water instead of gatorade. If the race takes her a long time, according to the model, what will be the eventual total mass of salt in her system?

$$\frac{dM}{dt} = -\overset{1}{\frac{0.3}{30}} M$$

$$= -\frac{1}{100} M$$

$$M(t) \to 0 \quad 1$$

6. Omar is a recovering heroine addict. As part of his recovery program, he uses a chemical patch on his arm that delivers methadone in an oscillatory manner to avoid the adaptive response associated with constant delivery. The equation for the concentration of methadone in Omar's blood stream is

$$M' = P(t) - kM$$

where $P(t) = \alpha(1 + \cos(\omega t))$ is the rate in mg/min at which the patch delivers methadone and k is a positive constant that is determined by how quickly the drug is metabolised.

(a) **[7 pts]** Find the general solution $M(t)$. Hint: This can be done using an integrating factor or the Method of Undetermined Coefficients. The latter approach is the simpler one in this case.

$$M' + kM = \alpha(1 + \cos\omega t)$$

$$M_h = De^{-kt} \quad 1$$

$$M_p = 1\ A + B\cos\omega t \overset{1}{+} C\sin\omega t$$

$$M_p' = -\omega B\sin\omega t + \omega C\cos\omega t$$

$$M_p' + kM = kA + (\omega C + kB)\cos\omega t + (-\omega B + kC)\sin\omega t$$

$$= \alpha + \alpha\cos\omega t$$

$$1 \quad kA = \alpha \qquad \omega C + kB = \alpha \qquad -\omega B + kC = 0$$

$$A = \frac{\alpha}{k} \qquad \frac{\omega^2 B}{k} + kB = \alpha \longleftarrow C = \frac{\omega B}{k}$$

$$B = \frac{\alpha}{\frac{\omega^2}{k} + k} = \frac{\alpha k}{\omega^2 + k^2} \quad 1$$

$$C = \frac{\alpha\omega}{\omega^2 + k^2} \quad 1$$

$$1 \quad M(t) = De^{-kt} + \frac{\alpha}{k} + \frac{\alpha k}{\omega^2 + k^2}\cos\omega t + \frac{\alpha\omega}{\omega^2 + k^2}\sin\omega t$$

(b) **[2 pt - modified]** What is the amplitude of the oscillatory part of the general solution? Your answer should depend on k, α and ω. Simplify your answer as much as possible.

$$\text{Amp} = \sqrt{\frac{\alpha^2 k^2}{(\omega^2+k^2)^2} + \frac{\alpha^2\omega^2}{(\omega^2+k^2)^2}} \quad 1$$

$$= \alpha\sqrt{\frac{k^2+\omega^2}{(\omega^2+k^2)^2}}$$

$$= \frac{\alpha}{\sqrt{\omega^2+k^2}} \quad 1$$

(c) **[2 pt]** Express the oscillatory part of the solution as a single cosine expression for the case of $\omega = k$?

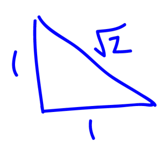
$$\frac{\alpha k}{\omega^2+k^2}\cos\omega t + \frac{\alpha\omega}{\omega^2+k^2}\sin\omega t$$

$$= \frac{\alpha}{\sqrt{\omega^2+k^2}}\left(\frac{k}{\sqrt{\omega^2+k^2}}\cos\omega t + \frac{\omega}{\sqrt{\omega^2+k^2}}\sin\omega t\right)$$

$$\cos(\omega t - \phi) = \cos\omega t\cos\phi + \sin\omega t \sin\phi$$

$$\cos\phi = \frac{k}{\sqrt{\omega^2+k^2}} = \frac{k}{\sqrt{2k^2}} = \frac{1}{\sqrt{2}} > 0$$

$$\sin\phi = \frac{1}{\sqrt{2}} > 0 \qquad \text{(so quad I)}$$



$$\phi = \frac{\pi}{4}$$

1 1

$$\frac{\alpha k}{\omega^2+k^2}\cos\omega t + \frac{\alpha\omega}{\omega^2+k^2}\sin\omega t = \boxed{\frac{\alpha}{\sqrt{\omega^2+k^2}}\cos\left(\omega t - \frac{\pi}{4}\right)}$$